THE CONCEPT OF INTEGRABILITY ON CANTOR SETS FOR HAMILTONIAN SYSTEMS*

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Abstract. Differentiable Hamiltonian systems close to nondegenerate, integrable Hamiltonian systems are shown to be integrable on a Cantor set in the sense that on some Cantor set, (i) the invariant KAM-tori form a smooth foliation, (ii) there exist smooth, independent integrals in involution, and (iii) there exists a complete solution of the Hamilton Jacobi equation. The complement of the Cantor set is shown to be small in measure.

(a) We are concerned with the question, to what extend perturbations of integrable Hamiltonian systems still resemble the unperturbed, integrable system, although integrability is generally lost and stochastic motion may appear in certain regions of the phase space. In more geometrical terms, we are concerned with the existence of a smooth foliation of invariant KAM-surfaces, filling the phase space with the exception of a set with small measure.

In the following we present the essence of our results. Some more technical details and proofs will appear elsewhere.

(b) Hamiltons equations of motion read:

$$\dot{q} = H_p(q, p), \qquad \dot{p} = -H_q(q, p), \tag{1}$$

where the dot indicates differentiation with respect to the time. Since integrable systems may be characterized by the existence of action-angle-variables, such that the Hamiltonian depends on the action variable alone, we write

$$H(q, p) = H^0(p) + \varepsilon H^1(q, p), \qquad \varepsilon \ll 1,$$

where H is assumed to have period 2π in each component of $q = (q_1, \dots, q_n)$, while $p = (p_1, \dots, p_n)$ varies over some domain I in R^n. The phase space therefore is

$$T^n \times I,$$

where T^n is the usual n-torus obtained by identifying points in R^n whose components differ by integer multiples of 2π; we assume $n \geqslant 2$.

For $\varepsilon = 0$ the system is governed by the integrable Hamiltonian H^0, and the equations of motion reduce to

$$\dot{q} = \omega, \qquad \dot{p} = 0$$

with

$$\omega = H_p^0(p).$$

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The whole phase space is completely foliated into an n-parameter family of invariant tori $T^n \times \{p\}$ with linear flow, so called *Kronecker systems* (T^n, ω). The components of ω provide integrals in involution for the motion of the system. In general, they are also functionally independent on I, which amounts to

$$\det \frac{\partial \omega}{\partial p} = \det H^0_{pp} \neq 0 \tag{2}$$

This is the so called *nondegenerate case*, which we will assume in the sequel. In other words, if the 'amplitudes' p vary over some open set, then the 'frequencies' ω vary over some open set, too, and it causes no loss of generality to require

$$H^0_p : I \to \Omega$$

to be actually a diffeomorphism between I and the set Ω of all frequencies ω of the integrable system. So we may as well parameterize the tori by their frequencies. Then different types of tori exist side by side, for example those with rational frequencies carrying periodic orbits, and those with rationally independent frequencies carrying transitive orbits. In fact, both types of tori form *dense* subsets in phase space.

We want to continue these tori for small $\varepsilon \neq 0$. First we observe that some sort of nondegeneracy condition like (2) for the integrable system is essential; if it is too degenerate, the motion may become ergodic on each energy surface, thus destroying *all* tori [3]. But even if the integrable system is nondegenerate, those tori with periodic orbits will generally break up under small perturbations, as was already known to Poincaré. Finally, by the work of Kolmogorov, Arnold and Moser, it turned out that those tori will persist whose frequencies are not only rationally independent, but satisfy a 'small divisor condition'

$$|(\omega, k)| \geqslant \gamma |k|^{-\tau}, \qquad 0 \neq k \in Z^n$$

with $\gamma > 0$. Simple measure theoretical considerations show that in fact almost all points in R^n satisfy such a condition for *some* γ while $\tau > n - 1$ is kept fixed; so we can find such points in the open set Ω, too. However, we can't allow γ to vary, but we have to *fix* it in advance, since it will enter in the smallness condition for the perturbation. Therefore we single out the *Cantor set*

$$\Omega_\gamma \subset \Omega$$

of those frequencies satisfying the small divisor condition for the given γ and having also distance $\geqslant \gamma$ to the boundary of Ω. Obviously

$$\Omega - \bigcup_{\gamma > 0} \Omega_\gamma$$

is a set of measure zero, so Ω_γ becomes large for small γ.

Now the results of Kolmogorov, Arnold and Moser assert the following [1, 4, 8, 10, 11]. Let the integrable Hamiltonian H^0 be real analytic and nondegenerate, and let the perturbed Hamiltonian $H = H^0 + \varepsilon H^1$ be of class C^r with $r > 2\tau + 2 > 2n$. Then there exists a positive, γ-independent δ, such that for

$$|\varepsilon| < \gamma^2 \delta$$

the perturbed system possesses invariant Kronecker systems (T^n, ω) for all $\omega \in \Omega_\gamma$, close to the corresponding unperturbed Kronecker systems.

Hence there exists a whole bunch of invariant tori parameterized over the Cantor set Ω_γ. But all these tori are constructed separately as subsystems of (1), and it is not at all clear to what extend they fit together in a smooth way. The only result in this direction was obtained by Arnold [1] for analytic perturbations where he showed the tori to depend continuously on ω.

Our main result states the existence of *one* coordinate system which straightens out *all* these invariant tori at the same time, thereby proving their foliation to be differentiable on Ω_γ. It also will enable us to estimate very easily the measure of their complement *in phase space* in the differentiable case which is not possible with the results known so far.

THEOREM: *Let the integrable Hamiltonian H^0 be real analytic and non-degenerate, such that the frequency map is a diffeomorphism $I \to \Omega$, and let the perturbed Hamiltonian $H = H^0 + \varepsilon H^1$ be of class $C^{\alpha\lambda + \lambda + \tau}$ with $\lambda > \tau + 1 > n$ and $\alpha > 1$.*

Then there exists a positive, γ-independent δ, such that for $|\varepsilon| < \gamma^2 \delta$ with γ sufficiently small, there exists a diffeomorphism

$$T : T^n \times \Omega \to T^n \times I$$

which on $T^n \times \Omega_\gamma$ transforms the Hamiltonian equations of motion into

$$\dot{\theta} = \omega, \qquad \dot{\omega} = 0. \tag{3}$$

T is of class C^α for noninteger α and close to the inverse of the frequency map; its Jacobian determinant is uniformly bounded from above and below.

In addition, if H is of class $C^{\beta\lambda + \lambda + \tau}$ with $\alpha \leqslant \beta \leqslant \infty$, then one can modify T outside $T^n \times \Omega_\gamma$ so that T is of class C^β for noninteger β.

Thus for each $\omega \in \Omega_\gamma$ the map $\theta \to T(\theta, \omega)$ parameterizes an invariant Kronecker system (T^n, ω), and this parameterization is differentiable in ω. This allows us to speak of an "integrable system on the Cantor set $T^n \times \Omega_\gamma$".

Actually, the transformation T is much smoother tangentially to the invariant tori than transversally to them, but we will suppress this aspect of 'anisotropic differentiability' here.

Of course, the transformation T is *not* symplectic. A similar 'normalization' *can* be achieved with a symplectic transformation, as we will see later on; in this case, however, the basis of the foliation can not be fixed in advance like Ω_γ, but has to be determined in dependence on the perturbation.

(c) We indicate the line of proof. We obtain T as the product of two transformations, namely the real analytic inverse Ψ_0 of the frequency map leaving the angle variables fixed, and a diffeomorphism Φ on $T^n \times \Omega$ close to the identity:

$$T = \Psi_0 \circ \Phi.$$

Ψ_0 is determined by the integrable system alone and is used to rephrase the problem in terms of the (θ, ω)-variables. It is only the diffeomorphism Φ, which depends on the perturbation, and its essential part is

$$\phi = \Phi | T^n \times \Omega_\gamma,$$

a map defined on a Cantor set. In fact, it is this map ϕ which we will construct first by the well known iteration process of Newton type due to Kolmogorov [4] and Arnold [1] combined with an approximation of the differentiable perturbation by real analytic ones, an idea due to Moser [7]. We will end up with a sequence of real analytic transformations ϕ_i, where *both* ϕ_i *and* $D\phi_i$ converge uniformly on $T^n \times \Omega_\gamma$:

$$\phi_i \to \phi, \qquad D\phi_i \to \phi'.$$

Now the point is to interpret this limit (ϕ, ϕ') as a *differentiable map* on the closed set $T^n \times \Omega_\gamma$ *in the sense of Whitney* [12, 14]. This allows us to extend ϕ to a map $\Phi : T^n \times \Omega \to T^n \times \Omega$ such that

$$\Phi | T^n \times \Omega_\gamma = \phi, \qquad D\Phi | T^n \times \Omega_\gamma = \phi'.$$

Higher differentiability is obtained similarly. However, these extensions are by no means unique, and depend in particular on the differentiability order. On the other hand, the map ϕ *is* unique up to a phase shift.

(d) There already exist some related results for mappings. For perturbations of twist maps in the plane preserving the intersection property Lazutkin [5] established the existence of a differentiable family of invariant curves parameterized over a Cantor set of rotation numbers. Svanidze [13] extended these results to higher dimensional mappings preserving a certain integral. They both use a modified version of Moser's original technique [6] where at each step of the iteration the equations are altered near finitely many resonances to obtain smooth global transformations Φ_i converging to some limit Φ. This technique, however, requires an excessive amount of differentiability in the perturbation; it also does not apply to the C^∞-case.

(e) There are two simple corollaries to our Theorem.

In view of (3) the components of ω surely form functionally independent integrals for the flow on $T^n \times \Omega_\gamma$. They are also in involution with respect to the transformed symplectic structure on $T^n \times \Omega_\gamma$. Expressing them as functions in q and p we obtain

COROLLARY 1: *On phase space there exist n smooth, independent functions, which are in involution and integrals of the motion when restricted to the image of* $T^n \times \Omega_\gamma$ *under T. They can be chosen to be of class* C^β *if the Hamiltonian is of class* $C^{\beta\lambda+\lambda+\tau}$ *with non-integer* β.

This further justifies to speak of an integrable system on a Cantor set. It generalizes a result of Chierchia and Gallavotti [2], who recently constructed such integrals in the analytic case.

Now we estimate the measure of the set $\mathscr{E}$ left out by the invariant tori in phase space in terms of the perturbation parameter ε. For small ε this measure will also be small. In the analytic case, this result is due to Arnold [1]; in the differentiable case, it has been stated frequently, but to our knowledge no proofs have been given so far.

The natural measure in this context is the invariant Liouville measure $(\Sigma\, dp_i \wedge dq_i)^n$, which happens to be the Lebesgue measure m. Now, by the change of variables formula and the uniform bounds for the Jacobian determinant of T, the measure of the full space $\mathscr{I} = T^n \times I$ is comparable with the measure of Ω, while $m(\mathscr{E})$ is comparable with $m(\Omega - \Omega_\gamma)$. But for a bounded set Ω with piecewise smooth boundary it is

$$m(\Omega - \Omega_\gamma) = 0(\gamma)\, m(\Omega). \tag{4}$$

Choosing γ proportional to $\sqrt{\varepsilon}$ we find

COROLLARY 2: *If* (4) *holds then the set of all invariant tori in phase space* $\mathscr{I}$ *leaves out a set* $\mathscr{E}$ *with*

$$m(\mathscr{E}) = 0(\sqrt{\varepsilon})\, m(\mathscr{I}),$$

where m *is the invariant Liouville measure associated with the symplectic structure, and* ε *is the perturbation parameter.*

We mention a third corollary which makes use of a more precise form of the smallness condition of the perturbation which we do not give here.

Consider an equilibrium in a sufficiently smooth Hamiltonian system with characteristic exponents

$$i\alpha_1, \ldots, i\alpha_n, \; -i\alpha_1, \ldots, -i\alpha_n$$

all on the imaginary axes. Assume there exist no resonances up to some finite order $\ell \geqslant 4$:

$$\Sigma k_j \alpha_j \neq 0 \quad \text{if} \quad 1 \leqslant |k_1| + \ldots + |k_n| \leqslant \ell. \tag{5}$$

Then one can introduce canonical coordinates (u, v) in some neighbourhood of the equilibrium such that the Hamiltonian is reduced to a Birkhoff normal form of degree ℓ up to terms of order $\ell + 1$. That is, we have

$$H = H^0 + \hat{H}$$

with a unique polynomial H^0 in $r_i = u_i^2 + v_i^2$ of order ℓ with respect to u and v, while $\hat{H}$ is small of order $\ell + 1$ in u and v. The polynomial part is integrable, and it is non-degenerate if

$$\det(H^0_{r_i r_j}) \neq 0. \tag{6}$$

If both (5) and (6) hold we speak of a general elliptic equilibrium of order ℓ.

COROLLARY 3: *Near a general elliptic equilibrium of order $\ell \geqslant 4$ in a sufficiently smooth Hamiltonian system one has*

$$\mathcal{m}(\mathscr{E}) = 0(r^{\lambda})\mathcal{m}(\mathscr{P}_r), \qquad \lambda = \frac{\ell - 3}{4}$$

for the set $\mathscr{E}$ of instability in the polydisc $\mathscr{P}_r : u_i^2 + v_i^2 < r$ around the equilibrium, provided the Hamiltonian is in Birkhoff normal form in (u, v) up to order $\ell + 1$.

(f) As a side product of the iteration process leading to the transformation T of our Theorem we also obtain on some Cantor set in phase space a solution of the Hamilton Jacobi equation

$$H(q, P - S_q(q, P)) = K(P). \tag{7}$$

Such a solution is called *complete*, if in addition

$$\det(I - S_{qP}) \neq 0.$$

For a real analytic Hamiltonian H, Poincaré [9] gave a solution in terms of a formal power series in a small perturbation parameter known as the Lindstedt series. However, these series are doomed to be divergent due to the occurrence of small divisors which vanish on a dense set. In fact, a complete solution of (7) on an open set would imply the integrability of the Hamiltonian system, which generically is not the case. Nonetheless, there indeed exist solutions for particular choices of P, and this is nothing but a restatement of the KAM-theorem on the existence of invariant tori. We assert that these particular solutions depend smoothly on P.

ADDENDUM TO THEOREM: *On $T^n \times I$ there exists a function S which is small with ε, and a nondegenerate Hamiltonian K, which is independent of the angle variables, such that*

$$H(q, P - S_q(q, P))\big| T^n \times I_\gamma = K(P), \tag{8}$$

where I_γ is the inverse image of Ω_γ under the map $P \to K_P(P)$. In addition, if H is of class $C^{\beta\lambda + 2\lambda}$, then one can modify S and K respectively outside $T^n \times I_\gamma$ to be of class C^β and $C^{\beta+2}$ for noninteger β; moreover, (8) may be differentiated as often as S allows.

Note that the differentiability of Equation (8) is not a matter of course, since I_γ is nowhere dense.

Thus, for a sufficiently smooth Hamiltonian and a sufficiently small perturbation, we obtain a complete solution of the Hamilton Jacobi equation on the Cantor set $T^n \times I_\gamma$. This allows us to derive again not only the existence of invariant tori, but also the linearity of the flow on each torus, since we may differentiate with respect to P.

We may even take S to define implicitly a *symplectic* map $(Q, P) \to (q, p)$ by

$$p = P - S_q(q, P), \qquad Q = q - S_p(q, P),$$

transforming the perturbed Hamiltonian H on $T^n \times I_\gamma$ into the integrable Hamiltonian K. This map is in fact *exact symplectic*, since S is 2π-periodic in $q_1, \ldots, q_n$.

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