July 21, 1996. Revised October 6, 1996
In: *Proceedings of the ECM-2, Budapest, 1996.*
Progress in Mathematics **169** (1998) 167–186

Nonlinear Partial Differential Equations, Birkhoff Normal Forms, and KAM Theory

JÜRGEN PÖSCHEL
Universität Stuttgart

Introduction

The purpose of this talk is twofold. First, I want to give another example of how tools and techniques which are well proven in the world of finite dimensional dynamical systems may be applied in the world of infinite dimensional evolution equations. In this case this tool is the so called Birkhoff normal form of Hamiltonian mechanics, which allows to view Hamiltonian systems near an equilibrium as small perturbations of integrable systems. In the infinite dimensional world, such a normal form allows us to view certain nonlinear evolution equations not only as small perturbations of integrable *partial* differential equation, but also as small perturbations of infinite dimensional, integrable *ordinary* differential equations. In addition, the calculations involved are rather elementary.

Second, such normal forms enable one to apply, in a rather effortless way, an infinite dimensional extension of the classical KAM theory and thus establish the existence of large families of time-quasi-periodic solutions which are linearly stable. This may help to explain the numerical observations of recurrent, non-ergodic behaviour for those evolution equations. Here, we continue work of Sergej Kuksin, which he described at the first ECM in Paris in 1992.

1 Nonlinear Schrödinger and wave equations

As a first example, consider the nonlinear Schrödinger equation

$$\mathrm{i}u_t = u_{xx} - mu - f(|u|^2)u \tag{1}$$

on the bounded interval $[0, \pi]$ with Dirichlet boundary conditions

$$u(0,t) = 0 = u(\pi, t), \qquad -\infty < t < \infty.$$

Here, m is a real parameter, and f is a real analytic function around 0 in $\mathbb{C}$ such that

$$f(|u|^2)u = a\,|u|^2\,u + \cdots, \qquad a \neq 0.$$

So after rescaling u we have

$$iu_t = u_{xx} - mu \mp |u|^2\,u + O_5(u).$$

This equation is Hamiltonian. As the phase space, take $H_0^1([0,\pi])$, the Sobolev space of all complex valued L^2-functions on $[0,\pi]$ with an L^2-derivative and vanishing boundary values. As Hamiltonian, take

$$H = \tfrac{1}{2}\,\langle Au, u\rangle + \tfrac{1}{2}\int_0^\pi F(|u|^2)\,\mathrm{d}x,$$

where $A = -\mathrm{d}^2/\mathrm{d}x^2 + m$, F is the primitive of f with, say, $F(0) = 0$, and $\langle u, v\rangle = \operatorname{Re}\int_0^\pi u\bar{v}\,\mathrm{d}x$. Then (1) is equivalent to

$$\dot{u} = \mathrm{i}\nabla H(u),$$

where the gradient is taken with respect to $\langle\cdot,\cdot\rangle$. The underlying symplectic structure is $[u, \tilde{u}] = \langle \mathrm{i}u, \tilde{u}\rangle$.

As a second example, consider the nonlinear wave equation

$$u_{tt} = u_{xx} - mu - f(u) \tag{2}$$

on the same bounded interval $[0,\pi]$ with Dirichlet boundary conditions. In this case, $m > 0$, and f is a real analytic, *odd* function such that

$$f(u) = au^3 + O_5(u), \qquad a \neq 0.$$

This class of equations comprises the sine-Gordon, the sinh-Gordon and the ϕ^4-equation, given by

$$mu + f(u) = \begin{cases} \sin u \\ \sinh u \\ u + u^3 \end{cases},$$

respectively, as well as small perturbations of them of order five and more.

In this case, the phase space may be taken as $H_0^1([0,\pi]) \times L^2([0,\pi])$ with real coordinates u and $v = u_t$. As Hamiltonian, take

$$H = \tfrac{1}{2}\langle v, v\rangle + \tfrac{1}{2}\langle Au, u\rangle + \int_0^\pi F(u)\,\mathrm{d}x,$$

where A and F are as above, and $\langle\cdot,\cdot\rangle$ is the standard L^2-product. Then (2) is equivalent to

$$\dot{u} = \frac{\partial H}{\partial v} = v, \qquad \dot{v} = -\frac{\partial H}{\partial u} = -Au - f(u).$$

The underlying symplectic structure is $[(u,v),(\tilde{u},\tilde{v})] = \langle u, \tilde{v}\rangle - \langle \tilde{u}, v\rangle$.

We are going to study solutions of these equations of small amplitude. In a first approximation this allows us to consider the nonlinear terms as small perturbations of the linear Schrödinger and the Klein-Gordon equation,

$$\mathrm{i}u_t = u_{xx} - mu \qquad \text{and} \qquad u_{tt} = u_{xx} - mu, \tag{3}$$

respectively. These are, of course, completely understood. Every solution is the superposition of the harmonic oscillations of their basic modes $\phi_k = \sin kx$, $k \geq 1$, which oscillate with *fixed* frequencies

$$\lambda_k = k^2 + m \qquad \text{and} \qquad \lambda_k = \sqrt{k^2 + m},$$

and arbitrary amplitudes. For instance,

$$u(x,t) = \sum_{k\geq 1} a_k e^{\mathrm{i}\lambda_k t}\phi_k(x), \qquad \sum_{k\geq 1} k^2 |a_k|^2 < \infty,$$

is the general solution in the Schrödinger case. The combined motion of the modes is periodic, quasi-periodic or almost-periodic, depending on whether one, finitely many, or infinitely many modes are excited. Moreover, the whole phase space is completely filled by these types of solutions. That is, the linear systems are completely integrable.

Upon restoring the nonlinear terms this foliation will not persist in its entirety, due to the long range coupling effects of the nonlinearities, and resonances, or near resonances, among the different modes. This applies even to the families of time-periodic solutions. One may only hope to prove the persistence of a *large portion* of it in the regime of small amplitude solutions, using some extension of KAM theory to infinite dimensional systems. Integrable linear systems, however, are totally *degenerate* from the point of view of KAM theory, as they show no frequency amplitude modulation whatsoever – all the frequencies are fixed. Hence, KAM theory is not applicable at this stage.

This difficulty may be circumvented by replacing the scalar parameter m by a potential $Q(x, \xi)$, which depends on sufficiently many external parameters $\xi = (\xi_1, \ldots, \xi_n)$. They allow to vary the frequencies of the basic modes and may thus substitute the usual nondegeneracy condition of KAM theory. As a result one obtains large Cantor sets of parameters for which there exist quasi-periodic solutions of small amplitude. This approach was taken by Wayne [35] and in some cases by Kuksin – see for example [22, 24, 25]. However, these Cantor sets do not include any open interval of constant potentials $Q \equiv m$ due to infinitely many nonresonance conditions imposed on the frequencies λ_k. Also, the number of parameters needed grows with the number of independent frequencies of the quasi-periodic solutions considered.

But linear systems are not the only integrable approximations that are available. There are also *nonlinear* integrable pde around. In the case of the Schrödinger equation this is the Zakharov-Shabat equation

$$\mathrm{i}u_t = u_{xx} - mu - |u|^2 u$$

on the real line with periodic boundary conditions [38]. In the case of the wave equation this is the sine-Gordon or the sinh-Gordon equation with periodic boundary conditions on the real line. So it is natural to consider (1) and (2) as perturbations of such nonlinear integrable pde's. This approach was taken by Kuksin and his collaborators – see [3, 6].

This approach, however, entails a lot of technical difficulties. First, a detailed knowledge of the solutions of these integrable pde is required, for example through explicit representations by theta-functions, to obtain the variational equations of motion along unperturbed quasi-periodic solutions and to bring these into constant coefficient form. Second, these coefficients have to satisfy certain nondegeneracy conditions in order to apply KAM, and this is quite difficult to verify. All this can be done, but the amount of work is formidable – see also [3, 4, 5, 6].

It is the purpose of this talk to describe an alternate approach via nonlinear integrable *ode*, which by comparison is short and elementary. This approach was first proposed in [29]. For the Schrödinger equation this was carried out by Kuksin and the author [28], and for the wave equation by the author [30]. The first step is to rewrite the nonlinear pde as an infinite dimensional system of ode's by introducing infinitely many coordinates. To simplify notations, we now focus our attention on the nonlinearities $\mp |u|^2 u$ and $\mp u^3$, respectively, since terms of higher order will not make any difference.

Using Fourier's classical approach we make for the Schrödinger equation the ansatz

$$u = \sum_{k \geq 1} q_k(t)\phi_k(x).$$

The time dependent coefficients are taken from some Hilbert space $\ell^s_{\mathbb{C}}$, $s \geq 1$, of complex sequences $q = (q_1, q_2, \dots)$ with $\sum_{k\geq 1} k^{2s} |q_k|^2 < \infty$. One then obtains a Hamiltonian system on $\ell^s_{\mathbb{C}}$ with Hamiltonian

$$\begin{aligned} H &= \Lambda + G \\ &= \tfrac{1}{2} \sum_{k\geq 1} \lambda_k |q_k|^2 \pm \tfrac{1}{4} \sum_{i,j,k,l} G_{ijkl} q_i q_j \bar{q}_k \bar{q}_l, \end{aligned} \tag{4}$$

where

$$\lambda_k = k^2 + m, \qquad G_{ijkl} = \int_0^\pi \phi_i \phi_j \phi_k \phi_l \, \mathrm{d}x,$$

and equations of motion

$$\dot{q}_k = 2\mathrm{i} \frac{\partial H}{\partial \bar{q}_k} = \mathrm{i}\lambda_k q_k + \dots, \qquad k \geq 1.$$

These are the classical Hamiltonian equations of motion for the real and imaginary parts of $q_k = x_k + \mathrm{i} y_k$ written in complex notation. The underlying symplectic structure is $\frac{\mathrm{i}}{2} \sum_k dq_k \wedge d\bar{q}_k$.

The quadratic term Λ describes the linear integrable Schrödinger equation and gives rise to a linear Hamiltonian vector field which is *unbounded* of order 2. The fourth order term G is not integrable, but gives rise to a *bounded* vector field on $\ell^s_{\mathbb{C}}$ of order 0.

The same approach to the wave equation requires only a bit more notation. Here we write

$$u = \sum_{k\geq 1} \frac{q_k(t)}{\sqrt{\lambda_k}} \phi_k(x), \qquad v = u_t = \sum_{k\geq 1} \sqrt{\lambda_k}\, p_k(t)\phi(x),$$

taking $(q, p) = (q_1, q_2, \ldots, p_1, p_2, \ldots)$ from some real Hilbert space $\ell^s \times \ell^s$ with $s \geq \frac{1}{2}$. One obtains the Hamiltonian

$$\begin{aligned} H &= \Lambda + G \\ &= \tfrac{1}{2} \sum_{k\geq 1} \lambda_k (p_k^2 + q_k^2) \pm \tfrac{1}{4} \sum_{i,j,k,l} G_{ijkl} q_i q_j q_k q_l, \end{aligned} \tag{5}$$

where

$$\lambda_k = \sqrt{k^2 + m}, \qquad G_{ijkl} = \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \int_0^\pi \phi_i \phi_j \phi_k \phi_l \,\mathrm{d}x,$$

and equations of motion

$$\dot{q}_k = \frac{\partial H}{\partial p_k} = \lambda_k p_k, \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k} = -\lambda_k q_k - \frac{\partial G}{\partial q_k}.$$

The underlying symplectic structure is $\sum_k dq_k \wedge dp_k$. In this case the linear integrable vector field is unbounded of order 1 on $\ell^s \times \ell^s$, whereas the nonlinear vector field is bounded of order -1.

In both cases we are now dealing with a Hamiltonian system of infinitely many degrees of freedom near an *elliptic* equilibrium located at the origin. In the classical theory, the standard tool to investigate such systems is their *Birkhoff normal form*, or its generalizations. Remarkably, such a normal form is available here without any further assumption. Moreover, its coefficients are easily determined.

To this end one first notices that in both cases the fourth order coefficients satisfy

$$G_{ijkl} = 0 \quad \Leftrightarrow \quad i \pm j \pm k \pm l \neq 0,$$

for some combination of plus and minus signs. Next, in the Schrödinger case, one has the following elementary nonresonance of order four among the frequencies at the equilibrium.

Lemma 1.1 ([28]) *If $i \pm j \pm k \pm l = 0$, but $\{i, j\} \neq \{k, l\}$, then for $\lambda_k = k^2 + m$ one has*

$$\lambda_i + \lambda_j - \lambda_k - \lambda_l = i^2 + j^2 - k^2 - l^2 \neq 0.$$

This allows us to define a convergent normalizing coordinate transformation in exactly the same way as in the classical case by comparison of coefficients, since these are the only divisors that show up [12]. Thus, from the Hamiltonian (4) we can remove all terms of order four which are not integrable. The upshot is the following result.

Theorem 1.2 ([28]) *There exists a real analytic, symplectic coordinate transformation Φ in a neighbourhood of the origin in $\ell^s_{\mathbb{C}}$ that for all m takes the Hamiltonian* (4) *of the nonlinear Schrödinger equation into its Birkhoff normal form of order four,*

$$H \circ \Phi = \frac{1}{2} \sum_{k \geq 1} \lambda_k \, |q_k|^2 + \frac{1}{2} \sum_{k,l \geq 1} Q_{kl} \, |q_k|^2 \, |q_l|^2 + O_6(q),$$

with uniquely determined coefficients

$$Q_{kl} = \frac{2 - \delta_{kl}}{2} G_{klkl} = \frac{4 - \delta_{kl}}{4\pi}.$$

The order of the corresponding nonlinear vector fields is still 0.

Thus, we obtain an infinite dimensional Hamiltonian system which is completely integrable up to order four in the classical sense.

The corresponding result for the wave equation is somewhat less complete due to asymptotic resonances among its frequencies λ_k. Here, the following estimate holds.

Lemma 1.3 ([30]) *If i, j, k, l are positive, not pairwise equal integers such that $i \pm j \pm k \pm l = 0$, then for $\lambda_k = \sqrt{k^2 + m}$ one has*

$$\left| \lambda_i \pm \lambda_j \pm \lambda_k \pm \lambda_l \right| \geq \frac{cm}{\sqrt{(n^2 + m)^3}}, \qquad n = \min(i, j, k, l),$$

for arbitrary combinations of plus and minus signs with some absolute constant c. Hence the left hand side is uniformly bounded away from zero on every compact m-interval in $(0, \infty)$.

As a consequence, we obtain the following partial normal form result. Let $z_k = \frac{1}{\sqrt{2}}(q_k + \mathrm{i}p_k)$ and $\bar{z}_k = \frac{1}{\sqrt{2}}(q_k - \mathrm{i}p_k)$. Then $z \in \ell^s_{\mathbb{C}}$.

Theorem 1.4 ([30]) *For each* $n \geq 1$ *and* $m > 0$ *there exists a real analytic, symplectic coordinate transformation* Φ_n *in some neighbourhood of the origin in* $\ell^s_{\mathbb{C}}$ *that takes the Hamiltonian* (5) *into*

$$H \circ \Phi_n = \sum_{k \geq 1} \lambda_k \, |z_k|^2 + \tfrac{1}{2} \sum_{\min(k,l) \leq n} Q_{kl} \, |z_k|^2 \, |z_l|^2 + O_4(z') + O_6(z),$$

with uniquely determined coefficients

$$Q_{kl} = 12(2 - \delta_{kl}) G_{klkl} = \frac{6}{\pi} \cdot \frac{4 - \delta_{kl}}{\lambda_k \lambda_l}$$

and $z' = (z_{n+1}, z_{n+2}, \ldots)$. *The order of the associated nonlinear vector fields is still* -1.

Moreover, the neighbourhood can be chosen uniformly for every compact m*-interval in* $(0, \infty)$, *and the dependence of* Φ_n *on* m *is real analytic.*

The first two terms of the normal form describe again an infinite dimensional integrable Hamiltonian system. Note that the interaction of the first n modes with *all* modes is normalized, but not the interaction of the higher modes with each other. Incidentally, it is possible to produce a complete normal form as in Theorem 1.2, but only at the expense of increasing the order of the nonlinear vector fields by 1. To such a normal form, the KAM theorem of section 4 below is *not* applicable.

We are now in a position to study the fate of quasi-periodic solutions, where only a finite number of modes is excited. Their analogue in the pde approach are the so called finite-gap-solutions. Suppose for simplicity that we want to excite the first n modes, $1 \leq n < \infty$. Introduce action-angle-coordinates for them by writing

$$q_k = z_k = \sqrt{I_k}\, e^{\mathrm{i}\varphi_k}, \qquad 1 \leq k \leq n.$$

In both cases we obtain a Hamiltonian $H = H_0 + H_1$ in coordinates (φ, I, z') in $\mathbb{T}^n \times \mathbb{R}^n \times \ell^s_{\mathbb{C}}$, where

$$H_0 = \sum_{1 \leq k \leq n} \lambda_k I_k + \tfrac{1}{2} \sum_{1 \leq k,l \leq n} Q_{kl} I_k I_l + \sum_{k \geq n+1} \lambda_k \, |z_k|^2 + \sum_{1 \leq k \leq n < l} Q_{kl} I_k \, |z_l|^2 \qquad (6)$$

and

$$H_1 = O_4(z') + O_6(z).$$

H_0 describes a *nonlinear* integrable Hamiltonian system with equations of motion

$$\begin{aligned}
\dot{\varphi}_k &= \lambda_k + \sum_{1\le l\le n} Q_{kl} I_l + \ldots, \\
\dot{I}_k &= 0, \\
\dot{z}_k &= \mathrm{i}\lambda_k z_k + \ldots.
\end{aligned}$$

There is an n-parameter family of invariant tori

$$\mathbb{T}^n \times \{I\} \times \{0\}, \qquad I \in \mathbb{R}^n_+,$$

carrying linear flows with amplitude-dependent frequencies, along which the variational equations of motion have constant coefficients Q_{kl} that are known explicitly. Note that this family is *different* from the analogous family for the quadratic Hamiltonian $\sum_k \lambda_k |z_k|^2$, where there is *no* frequency modulation.

To this Hamiltonian H_0 an infinite dimensional version of the KAM theorem given by Kuksin [25] or the author [31] can be applied. The nondegeneracy conditions for H_0 are easily checked, and H_1 may be regarded as a small perturbation of H_0, when the amplitudes are sufficiently small. As a result, for each n one obtains a Cantor family of time-quasi-periodic solutions of the nonlinear Schrödinger and wave equations with n independent frequencies, filling n-dimensional invariant tori in phase space which are linearly stable. The relative $2n$-dimensional Hausdorff measure of all these tori approaches 1 as one approaches the equilibrium $u \equiv 0$. For a more detailed and complete statement of the conclusions we refer to [28, 30].

To sum up, the key ingredients of this approach are the following. To get started the linear operator involved – in this case $-\mathrm{d}^2/\mathrm{d}x^2 + m$ with Dirichlet boundary conditions – has to have a pure point spectrum with a complete set of eigenfunctions. The point eigenvalues have to avoid certain low order resonances so that the transformation into Birkhoff normal form is defined. This, for example, makes it necessary that the cubic nonlinearities

$$\pm |u|^2 u \qquad \text{and} \qquad \pm u^3$$

do not depend explicitly on x; otherwise, there were non-integrable resonant terms which can not be transformed away for all parameter values of m. Their sign, however, is completely irrelevant. So it is not necessary to distinguish between the focusing and the defocusing cases.

The nonresonance conditions are reminiscent of Lyapunov's center theorem in the finite-dimensional theory: if, at an elliptic equilibrium of a real analytic Hamiltonian with characteristic frequencies $\lambda_1, \ldots, \lambda_n$, one has

$$\frac{\lambda_k}{\lambda_n} \notin \mathbb{Z}, \qquad 1 \leq k \leq n-1,$$

then there exists a disc through the equilibrium filled with periodic solutions with frequencies λ approaching λ_n as they approach the equilibrium. The nonresonance conditions are necessary: given a resonance there are nonlinearities so that such a family of periodic solutions does not exist [33].

For infinite dimensional systems there are similar results concerning the *non-persistence of breathers* under generic perturbations. See for example [16] and [34], respectively, for recent results concerning the sine-Gordon equation and certain classes of nonlinear wave equations. Very loosely speaking, the resonance occurs between the point eigenvalue of the unperturbed periodic solution – the breather – and the continuous spectrum of the unperturbed operator. These results indicate that to require a pure point spectrum not in low order resonance is not a technical shortcoming, but essential for persistence results of the kind above.

Some further restrictions arise from the requirements of the KAM theorem itself. First, the characteristic frequencies λ_k of the equilibrium have to grow at least linearly:

$$\lambda_k \sim k^d, \qquad d \geq 1.$$

This restricts the existence results essentially to problems with a 1-dimensional x-space. Second, the frequencies also have to be *simple*. This restricts one essentially to Sturm-Liouville boundary conditions, and excludes periodic boundary conditions, where eigenvalues are asymptotically double.

This restriction is more of a technical nature. Recently, Craig & Wayne [14] extended the Lyapunov center theorem to infinite dimensions and constructed Cantor discs of time-periodic solutions of a nonlinear wave equation with periodic boundary conditions. They used a Lyapunov-Schmidt reduction scheme together with small divisor estimates in the spirit of KAM theory. Their approach was then extended considerably by Bourgain, who obtained not only quasi-periodic solutions for Schrödinger equations [9], but also periodic and quasi-periodic solutions for some *two*-dimensional Schrödinger and wave equation [7, 8, 11]. With this technique, double eigenvalues are permitted. On the other hand, one does *not* obtain the linear stability of the solutions so constructed.

The results described so far concern the existence of quasi-periodic solutions filling finite-dimensional invariant tori in an infinite dimensional phase space. Nothing is known, however, about the existence of *almost-periodic* solutions. On one hand, the generic solution of the linear equations (3) are almost-periodic, with infinitely many excited modes and infinitely many independent frequencies. On the other hand, the nonlinearities effect a strong, long range coupling among all of them. This effect is beyond the control of the current techniques.

There are, however, existence results for simplified problems. Bourgain [10] considered the Schrödinger equation

$$\mathrm{i}u_t = u_{xx} - V(x)u - |u|^2\, u$$

on $[0, \pi]$ with Dirichlet boundary conditions, depending on some analytic potential V. Given an almost-periodic solution of the linear equation with *very* rapidly decreasing amplitudes and nonresonant frequencies, he showed that the potential V may be modified so that this solution persists for the nonlinear equation. Thus, the potential serves as an infinite dimensional parameter, which has to be chosen properly for each initial choice of amplitudes.This result is obtained by iterating the Lyapunov-Schmidt reduction introduced by Craig & Wayne.

A similar result was obtained independently by the author for the equation

$$\mathrm{i}u_t = u_{xx} - V(x)u - N(u),$$

where

$$N(u) = \Psi\big(f(|\Psi u|^2)\Psi u\big),$$

$\Psi\colon u \mapsto \psi * u$ a convolution operator which is smoothing of order greater than $1/4$. This smoothing makes it possible to *iterate* the KAM theorem about the existence of quasi-periodic solutions. As a result, one obtains for – in a suitable sense – almost all potentials V a set of almost periodic solutions, which – again in a suitable sense – has density one at the origin. See [32] for more details.

2 Perturbed KdV equations

Now we consider small Hamiltonian perturbations of the best known nonlinear integrable pde, the KdV equation. That is, we consider the equation

$$u_t = u_{xxx} - 6uu_x + \varepsilon\partial_x f(u) \tag{7}$$

with periodic boundary conditions,

$$u(x+1, t) = u(x, t),$$

where f is some real analytic function on the real line. For the simplicity of the exposition, we may restrict ourselves to the case of zero mean solutions,

$$[u] = \int_0^1 u(x,t)\,\mathrm{d}x = 0,$$

although the general case presents no additional difficulties.

The KdV equation belongs to a whole family of integrable pde. For instance, we may also consider perturbations of the second KdV equation,

$$u_t = u_{xxxxx} - 10uu_{xxx} - 20u_x u_{xx} + 30u^2 u_x + \varepsilon \partial_x f(u), \tag{8}$$

subject to the same boundary and mean value conditions. We will consider both equations simultaneously.

These equations are Hamiltonian. As a phase space take the Sobolev space $H_0^s(S^1)$, $s \geq 1$, of real valued functions on $\mathbb{R}$ with period 1 and mean value zero, endowed with the Poisson structure due to Gardner,

$$\{F, G\} = \int_0^1 \frac{\partial F}{\partial u} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial G}{\partial u}\,\mathrm{d}x.$$

The Hamiltonians corresponding to the above equations are then

$$H^1(u) = \int_0^1 \left(\tfrac{1}{2}u_x^2 + u^3 + \varepsilon F(u)\right)\mathrm{d}x$$

and

$$H^2(u) = \int_0^1 \left(\tfrac{1}{2}u_{xx}^2 + 5uu_x^2 + \tfrac{5}{2}u^4 + \varepsilon F(u)\right)\mathrm{d}x,$$

where F is the primitive of f with $F(0) = 0$. The above evolution equations are then equivalent to

$$\dot{u} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial H^\iota}{\partial u}$$

for $\iota = 1$ and $\iota = 2$, respectively.

Note that for the first KdV equation, the linear operator is unbounded of order 3, while the nonlinear operator is unbounded of order 1. For the second KdV equation, these numbers are 5 and 3, respectively. So, in contrast to the nonlinear Schrödinger and wave equations considered in section 1, also the nonlinearities are given by *unbounded* operators. This causes some complications in the KAM theory, as we will indicate below.

Results for the perturbed KdV equation were first obtained by Kuksin [23, 27]. He showed that the so called n-gap solutions persist for sufficiently small ε, where the bound on ε depends on the solution considered, in particular its number of gaps. To apply the KAM theory, however, he had to go a long way. First, the Its-Matveev formula for periodic solutions and the Zakharov-Akhiezer-Ψ-functions were needed to obtain the variational equations of motion along n-gap solutions and to put them into constant coefficient form. Then, further investigations based on a beautiful result of Krichever [21] were needed to verify the nondegeneracy and nonresonance conditions required by the KAM theorem.

The use of Birkhoff normal forms provides an alternate, elementary approach to the same results, at least for *small* solutions. The following presents joint work of Kappeler and the author [20]. Write

$$u = \sum_{k \neq 0} \gamma_k q_k e^{2\pi \mathrm{i} kx} = \Psi q \tag{9}$$

with

$$\gamma_k = \sqrt{2\pi\, |k|}, \qquad \bar{q}_k = q_{-k},$$

and $q = (q_1, q_2, \dots) \in \ell_{\mathbb{C}}^{s+1/2}$. The KdV Hamiltonian for $\varepsilon = 0$ in these new coordinates is

$$\begin{aligned} H^1(q) &= \Lambda + G \\ &= \sum_{k \geq 1} (2\pi k)^3 \, |q_k|^2 + \sum_{k+l+m=0} \gamma_k \gamma_l \gamma_m q_k q_l q_m . \end{aligned}$$

The equations of motion are the classical ones in complex coordinates:

$$\dot{q}_k = \mathrm{i} \frac{\partial H^1}{\partial \bar{q}_k}, \qquad k \geq 1.$$

The second KdV Hamiltonian is also easy to write down. It just contains more terms and will be given below in normal form.

To transform the first Hamiltonian into its Birkhoff normal form, two coordinate transformations are required, one to eliminate the cubic term, and one to normalize the resulting fourth order term. Both calculations are elementary. The pertinent nonresonance conditions are satisfied, since first, no three cubics add up to zero, and second, an analogue of Lemma 1.1 for four cubics holds. The result is the following.

Theorem 2.1 ([20]) *There exists a real analytic, symplectic coordinate transformation* Φ *in a neighbourhood of the origin in* $\ell_{\mathbb{C}}^{s+1/2}$*, which transforms the first KdV Hamiltonian into*

$$H^1 \circ \Phi = \sum_{k \geq 1} (2\pi k)^3 \, |q_k|^2 - 3 \sum_{k \geq 1} |q_k|^4 + O_5(q)$$

and preserves the order of the nonlinear vector fields. This transformation satisfies $\Phi(0) = 0$, $d_0 \Phi = id$.

Note that the quartic term is in diagonal form.

The Hamiltonians of the KdV family are all in involution with each other. As a consequence, this transformation Φ not only normalizes the first KdV Hamiltonian up to order four, but indeed simultaneously normalizes *every other* KdV Hamiltonian up to order four as well. This considerably simplifies the calculations. To find the normal form of the second KdV Hamiltonian, we only have to calculate *one* Poisson bracket, since the coefficients of Φ are known already. Compare also Ito [18] for a sharper version in finite dimensions.

Theorem 2.2 ([20]) *The transformation* Φ *of the above theorem transforms every Hamiltonian of the KdV family into its Birkhoff normal form up to order four, preserving the order of the nonlinear vector fields. In particular, for the second KdV Hamiltonian one finds*

$$H^2 \circ \Phi = \sum_{k \geq 1} (2\pi k)^5 \, |q_k|^2 + \sum_{k,l \geq 1} Q_{kl} \, |q_k|^2 \, |q_l|^2 + O_5(q)$$

with uniquely determined coefficients

$$Q_{kl} = 20\pi^2 (6 - 7\delta_{kl}) \, kl.$$

These normal forms are sufficient to prove the persistence of quasi-periodic solutions of sufficiently small amplitude under small Hamiltonian perturbations of the equations as given in (7) and (8). Moreover, if the perturbing terms are of order three or more in u, no extra parameter ε is needed to make the perturbing terms small. The nonresonance and nondegeneracy conditions are easily checked with the explicit coefficients at hand, so that the KAM theorem as given below can be applied in the usual way.

More is needed, however, to study large solutions as well. Here, the global action-angle coordinates constructed by Kappeler *et al.* [1, 2, 19] give a very convenient handle. In our context, their result can be formulated as follows.

Theorem 2.3 ([1, 2, 19]) *There exists a real analytic, symplectic diffeomorphism*

$$\Omega\colon\ \ \ell^{3/2}_{\mathbb{C}} \to H^1_0(S^1), \quad q \mapsto u,$$

with a real analytic inverse, such that the KdV Hamiltonian (7) *is transformed into a function of the amplitudes* $|q|^2 = (|q_1|^2, \dots)$ *alone:*

$$H^1 \circ \Omega = \hat{H}^1(|q|^2).$$

The same holds for every other Hamiltonian in the KdV family.

This transformation satisfies

$$\Omega(0) = 0, \qquad d_0\Omega = \Psi,$$

where Ψ *is the transformation in* (9)*. Moreover, the restriction of* Ω *to each subspace* $\ell^{s+1/2}_{\mathbb{C}}$*,* $s \geq 1$*, is a bianalytic diffeomorphism between* $\ell^{s+1/2}_{\mathbb{C}}$ *and* $H^s_0(S^1)$*.*

We may view Ω as a global transformation taking H^1 into a complete global Birkhoff normal form, whereas our simple transformation $\Psi \circ \Phi$ is a local transformation into a local Birkhoff normal form of order four only. Both transformations, however, have the *same* linearization at the origin. The uniqueness of the Birkhoff normal form then implies that the two normal forms must agree up to order four. In other words, the local results provides us with the first terms of the Taylor series expansion of the globally integrable Hamiltonians. In the following we drop the $\hat{\ }$.

Theorem 2.4 ([20]) Ω *transforms the first and second KdV Hamiltonians into the globally integrable Hamiltonians*

$$H^\iota(|q|^2) = \sum_{k\geq 1} \lambda^\iota_k\, |q_k|^2 + \tfrac{1}{2} \sum_{k,l\geq 1} Q^\iota_{kl}\, |q_k|^2\, |q_l|^2 + \dots,$$

where

$$\lambda^1_k = (2\pi k)^3, \qquad Q^1_{kl} = -6\delta_{kl},$$

and

$$\lambda^2_k = (2\pi k)^5, \qquad Q^2_{kl} = 40\pi^2(6 - 7\delta_{kl})kl.$$

Now we are in a position to study perturbations of large n-gap solutions. Introducing the usual action-angle coordinates

$$q_k = \sqrt{I_k}\, e^{\mathrm{i}\varphi_k}, \qquad 1 \le k \le n,$$

we obtain an integrable system with Hamiltonian

$$H^\iota = \sum_{1\le k\le n} \lambda_k^\iota I_k + \frac{1}{2} \sum_{1\le k,l\le n} Q_{kl}^\iota I_k I_l + \sum_{k\ge n+1} \lambda_k^\iota \,|q_k|^2 + \sum_{1\le k\le n<l} Q_{kl}^\iota I_k \,|q_l|^2 + \cdots \tag{10}$$

and equations of motion

$$\dot\varphi_k = \omega_k^\iota, \qquad \dot I_k = 0, \qquad \dot q_k = \mathrm{i}\omega_k^\iota q_k,$$

where the frequencies ω_k^ι are real analytic functions of I and $|q'|^2 = (|q_{n+1}|^2, \dots)$ with expansion

$$\omega_k^\iota(I, |q'|^2) = \lambda_k^\iota + \sum_{1\le l\le n} Q_{kl}^\iota I_l + \cdots .$$

The dots stand for terms of higher order in I and at least first order in $|q'|^2$.

Thus, the variational equations along each invariant torus $\mathbb{T}^n \times \{I\} \times \{0\}$ have constant coefficients, which are real analytic functions of the position I of the torus. The relevant nonresonance and nondegeneracy conditions are easily checked at $I = 0$. Hence, by analyticity and local finiteness, they are satisfied on a dense open subset of the phase space. We may thus apply KAM theory to obtain the persistence of almost all finite gap solutions of the first and second KdV equation under sufficiently small Hamiltonian perturbations of the equations as given in (7) and (8).

Of course, the admissible size of the perturbations depends on the solution considered. For a fixed size of ε, a Cantor set of quasi-periodic solutions is obtained, which asymptotically fills the phase space as ε tends to zero.

To sum up, in the Hamiltonian perturbation theory of the KdV family, the Birkhoff normal form serves two purposes. First, it provides an easy way to apply KAM theory to small solutions. Second – and this is probably more important – in connection with the global coordinates of Kappeler *et al.* it provides a sufficient control over the variational equations to verify the nonresonance and nondegeneracy conditions almost everywhere. So for this purpose the deep and intricate results of Krichever are not needed.

3 Water waves

So far we considered equations which could be understood as small perturbations of some nonlinear integrable pde. Now I want to describe another application of the Birkhoff normal form where seemingly there is no integrable pde in the background. This application is due to Zakharov [37] and Craig & Worfolk [15, 13], and I will merely report their results.

Consider the motion of an incompressible, irrotational ideal fluid in a two-dimensional domain of infinite depth,

$$S(\eta) = \{(x, y) : 0 < x < 2\pi, \ \ -\infty < y < \eta(x, t)\}.$$

There is a velocity potential φ, so that on this domain the velocities are

$$u = \nabla\varphi, \qquad \Delta\varphi = 0. \tag{11}$$

The boundary condition are

$$\varphi_y \to 0 \qquad \text{as} \qquad y \to -\infty$$

at the bottom and periodic at the lateral sides,

$$\eta(x + 2\pi, t) = \eta(x, t), \qquad \varphi(x + 2\pi, t) = \varphi(x, t).$$

The nonlinearity and time dependence of the problem are introduced by the boundary condition at the free top surface,

$$\eta_t = \varphi_y - \eta_x\varphi_x, \qquad \varphi_t = -g\eta - \tfrac{1}{2}\langle\nabla\varphi, \nabla\varphi\rangle,$$

where g is the gravitational constant.

These equations can be written in Hamiltonian form. The Hamiltonian is the total energy of the system. The appropriate choice of canonically conjugate variables was given by Zakharov [36] as

$$\eta(x) \qquad \text{and} \qquad \xi(x) = \varphi(x, \eta(x)),$$

the free surface and the trace of the potential on it. This trace together with the boundary conditions at the bottom and at the sides uniquely determines the harmonic potential φ on the domain $S(\eta)$. The Hamiltonian then turns out to be

$$\begin{aligned} H(\xi, \eta) &= K + V \\ &= \tfrac{1}{2} \int_0^{2\pi} \xi G(\eta)\xi \, \mathrm{d}x + \tfrac{1}{2} \int_0^{2\pi} g\eta^2 \, \mathrm{d}x. \end{aligned}$$

Here, $G(\eta)$ is the Dirichlet-Neumann operator which associates with ξ the normal derivative of the harmonic solution φ with $\varphi(x, \eta(x)) = \xi(x)$. The equations of motion are now

$$\dot{\eta} = \frac{\partial H}{\partial \xi} = G(\eta)\xi, \qquad \dot{\xi} = -\frac{\partial H}{\partial \eta} = -g\eta - \frac{\partial K}{\partial \eta}.$$

In the following we may restrict ourselves to zero mean functions,

$$[\eta] = 0, \qquad [\xi] = 0,$$

since these means are conserved quantities.

The Dirichlet-Neumann operator G is analytic in η, so it admits an expansion $G = G_0 + G_1 + \ldots$ into homogeneous terms in η. Consequently, there is a similar expansion of the Hamiltonian H,

$$H = H_2 + H_3 + \cdots,$$

where H_j is homogeneous of degree j in ξ and η. Thus, 0 is an equilibrium, which agrees well with the fact that the flat surface is a steady state of our system. In particular,

$$H_2 = \tfrac{1}{2} \int_0^{2\pi} (\xi G_0 \xi + g\eta^2) \, \mathrm{d}x.$$

Moreover, at infinite depth one simply has $G_0 = -\mathrm{i}\partial_x$. Expanding ξ and η into Fourier series with complex coefficients ξ_k and η_k and no constant terms, one obtains

$$H_2 = \sum_{k \geq 1} (k\, |\xi_k|^2 + g\, |\eta_k|^2).$$

Thus, 0 is an elliptic equilibrium with characteristic frequencies

$$\omega_k = \sqrt{gk}, \qquad k \geq 1.$$

This is the well known frequency dispersion for linear water waves.

Now consider the next terms of the Hamiltonian H. They can be written down explicitly, either by hand or by computer, but their exact form is not relevant here. As before, the cubic term H_3 contains only monomials in ξ_k and η_k whose indices add up to zero. They turn out to be nonresonant, since similarly to Lemma 1.1 one has

$$\omega_k \pm \omega_l \pm \omega_m \neq 0 \qquad \text{if} \qquad k \pm l \pm m = 0.$$

So formally they can be transformed away. Craig & Worfolk [15] showed that this formal transformation is indeed convergent.

Theorem 3.1 ([15]) *There exists a real analytic, symplectic coordinate transformation between small neighbourhoods of the origin in some suitably defined Hilbert space of exponentially decreasing sequences of Fourier coefficients such that*

$$H \circ \Phi = H_2 + \tilde{H}_4 + \tilde{H}_5 + \cdots .$$

That is, the third order term is completely removed.

The fourth order term, however, is more troublesome. There are not only the classical, *benign* resonances of order four of the form

$$\omega_k + \omega_l - \omega_k - \omega_l = 0,$$

which are not harmful, since they give rise to *integrable* terms in the normal form. There are also *non-benign* resonances such as

$$\omega_1 - \omega_4 - \omega_4 + \omega_9 = 0.$$

Indeed, all the non-benign resonances are easily classified – they are of this form with indices

$$pm^2,\ p(m+1)^2,\ pm^2(m+1)^2,\ p(m^2+m+1)^2,$$

with positive integers p and m. In the original Hamiltonian H the corresponding coefficients are *not* all zero. But then, a little miracle happens – which was first observed by Zakharov [37] and reported in Dyachenko & Zakharov [17].

Theorem 3.2 ([17, 15]) *In the transformed Hamiltonian $H \circ \Phi$ of the preceding theorem* all *the non-benign coefficients of $\tilde{H}_4$* vanish. *Hence, there exists a formal symplectic coordinate transformation Ψ that takes $H \circ \Phi$ into an integrable Birkhoff normal form up to order four.*

It has not been shown yet, however, that this formal transformation is convergent in some suitable sense.

If a statement is true for $n = 3$ and $n = 4$, it is customary to conjecture that it holds for all n. Therefore, Dyachenko and Zakharov [17] conjectured that the Birkhoff normal form of H is integrable to every order. Craig & Worfolk [15] showed, however, that this is already false at $n = 5$. The transformed Hamiltonian contains nonvanishing resonant terms of order five, which can not be transformed away. This makes it very likely that the whole Hamiltonian is not integrable.

To sum up, the Hamiltonian of deep water waves has the following properties. Due to some unexpected cancellation of terms, there are formal symplectic coordinates around the equilibrium, in which the Hamiltonian is in Birkhoff normal form up to order four. So, modulo their analyticity, the system may be viewed as a small perturbation of an infinite dimensional system of nonlinear integrable ode's. This integrability can not be pushed to higher order, due to resonant terms of order five.

Unfortunately, the present KAM theories do *not* apply to this Hamiltonian, even when it is proven to be analytic. Its characteristic frequencies are

$$\omega_k \sim \sqrt{k},$$

whereas for the KAM theories,

$$\omega_k \sim k$$

is the limiting case.

4 The basic KAM theorem

For the sake of completeness and reference we include a version of the infinite dimensional KAM theorem that is applicable to the Hamiltonians in section 1 and 2.

To study perturbations of the Hamiltonians (6) and (10) it is convenient to introduce local coordinates

$$\varphi = x, \qquad I = \xi + y$$

around each torus $\mathbb{T}^n \times \{\xi\} \times \{0\}$. We obtain a *family* of Hamiltonians

$$H_0 = \sum_{1 \le k \le n} \omega_k(\xi) y_k + \sum_{k \ge n+1} \omega_k(\xi) \, |q_k|^2 + \cdots$$

with

$$\omega_k(\xi) = \lambda_k + \sum_{1 \le l \le n} Q_{kl} \xi_l,$$

for which we want to prove the persistence of the invariant torus $\mathbb{T}^n \times \{0\} \times \{0\}$.

It turns out that the terms comprised in the dots may also be regarded as perturbations. So, we now consider a slightly more general family of simple Hamiltonians

$$H_0 = \sum_{1 \le k \le n} \omega_k(\xi) y_k + \sum_{k \ge n+1} \Omega_k(\xi)\, |q_k|^2$$

on the phase space $\mathbb{T}^n \times \mathbb{R}^n \times \ell^s_{\mathbb{C}}$, where the frequencies $\omega = (\omega_1, \dots, \omega_n)$ and $\Omega = (\Omega_{n+1}, \Omega_{n+2}, \dots)$ depend on n parameters

$$\xi \in \Pi \subset \mathbb{R}^n.$$

The parameter domain Π may be any *closed* subset of $\mathbb{R}^n$ of positive Lebesgue measure. For instance, Π may be a Cantor set.

For each parameter value ξ, there is an invariant torus $\mathbb{T}^n \times \{0\} \times \{0\}$ with linear flow, which is determined by the "internal" frequencies $\omega(\xi)$. In its normal space described by the q-coordinates, the origin is an elliptic equilibrium with "external" frequencies $\Omega(\xi)$. The aim is to prove the persistence of a large portion of this family under small perturbations $H = H_0 + H_1$ of H_0. To this end the following assumptions are made.

A - *Spectral Asymptotics.* There exists $d > 1$ and $\beta > 1$ such that

$$\Omega_k(\xi) = k^d + \dots + \tilde{\Omega}_k(\xi),$$

where the dots stand for fixed lower order terms not depending on ξ, and where the weighted functions

$$\frac{\tilde{\Omega}_k(\xi)}{k^{d-\beta}}$$

are *uniformly* Lipschitz on Π.

We assume here that $d > 1$. The case $d = 1$ may also be handled – see [25, 31] – but is somewhat more involved and not relevant here. We therefore omit it for the sake of brevity.

B - *Nondegeneracy.* The map

$$\xi \mapsto \omega(\xi)$$

is a homeomorphism between Π and its image which is Lipschitz in both directions.

Moreover, for all $(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty$ with $1 \le \sum_i |l_i| \le 2$ the resonance sets

$$R_{kl} = \{\xi \in \Pi : \langle k, \omega(\xi)\rangle + \langle l, \Omega(\xi)\rangle = 0\}$$

have Lebesgue measure zero.

We note that assumption A implies that R_{kl} is *empty* for *almost all* k and l, so this assumption together with assumption A concerns only finitely many k and l.

C - *Regularity.* The perturbation H_1 is real analytic in the space coordinates and Lipschitz in the parameters, and for each ξ its Hamiltonian vector field is a map

$$X_{H_1} : \mathbb{T}^n \times \mathbb{R}^n \times \ell^s_{\mathbb{C}} \supset D \to \mathbb{T}^n \times \mathbb{R}^n \times \ell^{s-\sigma}_{\mathbb{C}}$$

with

$$\sigma < d - 1,$$

where D is some neighbourhood of $\mathbb{T}^n \times \{0\} \times \{0\}$.

To formulate a smallness condition for the perturbations we introduce the complex domains

$$D(w,r) : \ |\mathrm{Im}\, x| < w, \ |y| < r^2, \ \|q\|_s < r$$

and the weighted vector field norms

$$\|X_H\|_r = |\partial_y H| + r^{-2} |\partial_x H| + r^{-1} \|\partial_{\bar q} H\|_{s-\sigma} .$$

Then, $\|X_H\|_{r,D(w,r)\times\Pi}$ denotes the supremum of $\|X_H\|_r$ over the set $D(w,r)\times\Pi$. Similarly, $\|X_H\|^L_{r,D(w,r)\times\Pi}$ denotes the associated Lipschitz semi-norm with respect to the parameters ξ over Π.

Theorem 4.1 ([20, 25, 27, 31]) *Suppose $H = H_0 + H_1$ satisfies assumptions A, B and C. If*

$$\varepsilon = \left\| X_{H_1} \right\|_{r,D(w,r)\times\Pi} + \left\| X_{H_1} \right\|^L_{r,D(w,r)\times\Pi}$$

is sufficiently small, then there exists

(a) *a Cantor set $\Pi' \subset \Pi$ such that the Lebesgue measure of $\Pi \setminus \Pi'$ tends to zero with ε;*

(b) *a Lipschitz continuous family of real analytic, symplectic coordinate transformations*

$$\Psi \colon \mathbb{T}^n \times \mathbb{R}^n \times \ell^s_{\mathbb{C}} \times \Pi' \to \mathbb{T}^n \times \mathbb{R}^n \times \ell^s_{\mathbb{C}}$$

close to the identity; and

(c) *perturbed frequencies* ω' *and* Ω' *close to the unperturbed ones, which are Lipschitz on* Π',

such that

$$H \circ \Psi = \sum_{1 \le k \le n} \omega'_k(\xi) y_k + \sum_{k \ge n+1} \Omega'_k(\xi) \, |q_k|^2 + \cdots ,$$

where the dots stand for higher order terms in y *and* q*. Hence, for each* $\xi \in \Pi'$ *the perturbed Hamiltonian admits a linearly stable invariant* n*-torus with internal frequencies* $\omega'(\xi)$ *and external frequencies* $\Omega'(\xi)$.

For a more precise version of this theorem see [31] and [20]. Note that its "output" is of the same form as its "input". This makes it possible to iterate the theorem to construct almost-periodic solutions for certain Hamiltonians depending on infinitely many parameters [32].

The first versions of this KAM theorem [25, 31] had to impose the additional hypotheses

$$\sigma \le 0,$$

that is, the perturbing vector fields had to be *bounded* as operators. This was sufficient for perturbing Schrödinger and wave equations, but not sufficient for the KdV equations, where the operators are *unbounded*. Strictly speaking, the proof of Kuksin's result [23] contained a gap, which he filled only recently in [27]. It required a very delicate estimate of a linear small divisor equation with *large, variable* coefficients. For another version, see also [20].

References

[1] D. Bättig, A.M. Bloch, J.-C. Guillot, & T. Kappeler, On the symplectic structure of the phase space for periodic KdV, Toda, and defocusing NLS. *Duke Math. J.* **79** (1995), 549–604.

[2] D. Bättig, T. Kappeler, & B. Mityagin, On the Korteweg-de Vries equation: Convergent Birkhoff normal form. *J. Funct. Anal.* **140** (1996), 335–358.

[3] R.F. BIKBAEV & S.B. KUKSIN, A periodic boundary value problem for the Sine-Gordon equation, small Hamiltonian perturbations of it, and KAM-deformations of finite-gap tori. *Algebra i Analis* **4** (1992) [Russian]. English translation in *St. Petersburg Math. J.* **4** (1993), 439–468.

[4] R.F. BIKBAEV & S.B. KUKSIN, On the parametrization of finite-gap solutions by frequency vector and wave-number vector and a theorem of I. Krichever. *Lett. Math. Phys.* **28** (1993), 115–122.

[5] A. BOBENKO & S.B. KUKSIN, Finite-gap periodic solutions of the KdV equation are non-degenerate. *Physics Letters* **161** (1991), 274–276.

[6] A. BOBENKO & S.B. KUKSIN, The nonlinear Klein-Gordon equation on an interval as a perturbed sine-Gordon equation. *Comment. Math. Helv.* **70** (1995), 63–112.

[7] J. BOURGAIN, Periodic solutions of hamiltonian perturbations of linear Schrödinger equations in higher dimensions. Preprint (1994).

[8] J. BOURGAIN, Quasi-periodic solutions of hamiltonian perturbations of 2D linear Schrödinger equations. Preprint IHES (1994).

[9] J. BOURGAIN, Construction of quasi-periodic solutions for hamiltonian perturbations of linear equations and applications to nonlinear pde. *Intern. Math. Res. Not.* **11** (1994), 475–497.

[10] J. BOURGAIN, Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations. *Geom. Funct. Anal.* **6** (1996), 201–230.

[11] J. BOURGAIN, Construction of periodic solutions of nonlinear wave equations in higher dimension. *Geom. Funct. Anal.* **5** (1995), 629–639.

[12] R.C. CHURCHILL, M. KUMMER, & D.L. ROD, On averaging, reduction and symmetry in Hamiltonian systems. *J. Diff. Equations* **49** (1983), 359–414.

[13] W. CRAIG, Birkhoff normal forms for water waves. *Contemporary Mathematics* (1996) (to appear).

[14] W. CRAIG & C.E. WAYNE, Newton's method and periodic solutions of nonlinear wave equations. *Commun. Pure Appl. Math.* **46** (1993), 1409–1498.

[15] W. CRAIG & P.A. WORFOLK, An integrable normal form for water waves at infinite depth. *Physica D* **84** (1995), 513–531.

[16] J. DENZLER, Nonpersistence of breather families for the perturbed sine Gordon equation. *Commun. Math. Phys.* **158** (1993), 397–430.

[17] A.I. DYACHENKO & V.E. ZAKHAROV, Is free-surface hydrodynamics an integrable system? *Phys. Lett. A* **190** (1994), 144–148.

[18] H. ITO, Convergence of Birkhoff normal forms for integrable systems. *Comment. Math. Helvetici* **64** (1989), 412–461.

[19] T. KAPPELER, Fibration of the phase space for the Korteweg-de Vries equation. *Ann. Inst. Fourier* **41** (1991), 539–575.

[20] T. KAPPELER & J. PÖSCHEL, Perturbations of KdV equations. In preparation.

[21] I. KRICHEVER, "Hessians" of integrals of the Korteweg-de Vries equation and perturbations of finite-gap solutions. *Funktional. Anal. i Prilozhen.* **21** (1987), 22–37. English translation in *Soviet Math. Dokl.* **27** (1983).

[22] S.B. KUKSIN, Hamiltonian perturbations of infinite dimensional linear systems with an imaginary spectrum. *Funktsional. Anal. Prilozhen.* **21** (1987), 22–37 [Russian]. English translation in *Funct. Anal. Appl.* **21** (1987), 192–205.

[23] S.B. KUKSIN, Perturbation theory for quasiperiodic solutions of infinite-dimensional Hamiltonian systems, and its application to the Korteweg-de Vries equation. *Matem. Sbornik* **136** (1988) [Russian]. English translation in *Math. USSR Sbornik* **64** (1989), 397–413.

[24] S.B. KUKSIN, Conservative perturbations of infinite dimensional linear systems depending on a vector parameter. *Funct. Anal. Appl.* **23** (1989), 62–63.

[25] S.B. KUKSIN, *Nearly integrable infinite-dimensional Hamiltonian systems.* Lecture Notes in Mathematics 1556, Springer, 1993.

[26] S.B. KUKSIN, KAM-theory for partial differential equations. In *Proceedings First European Congress of Mathematics* (Paris 1992), Birkhäuser, Basel, 1994, 123–157.

[27] S.B. KUKSIN, A KAM theorem for equations of the Korteweg-de Vries type. Preprint (1996).

[28] S.B. KUKSIN & J. PÖSCHEL, Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation. *Ann. Math.* **143** (1996), 149–179.

[29] J. PÖSCHEL, On elliptic lower dimensional tori in Hamiltonian systems. *Math. Z.* **202** (1989), 559–608.

[30] J. PÖSCHEL, Quasi-periodic solutions for a nonlinear wave equation. *Comment. Math. Helv.* **71** (1996), 269–296.

[31] J. PÖSCHEL, A KAM-theorem for some nonlinear partial differential equations. *Ann. Sc. Norm. Sup Pisa* **23** (1996), 119–148.

[32] J. PÖSCHEL, On the construction of almost periodic solutions for a nonlinear Schrödinger equation. In *Hamiltonian Systems with Three or More Degrees of Freedom*, C. Simo (Ed), Kluwer, Dordrecht (to appear)

[33] C.L. SIEGEL & J. MOSER, *Lectures on Celestial Mechanics.* Grundlehren 187, Springer, Berlin, 1971.

[34] I.M. SIGAL, Non-linear wave and Schrödinger equations I. Instability of periodic and quasiperiodic solutions. *Commun. Math. Phys.* **153** (1993), 297–320.

[35] C.E. WAYNE, Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory. *Commun. Math. Phys.* **127** (1990), 479–528.

[36] V.E. ZAKHAROV, Stability of periodic waves of finite amplitude on the surface of deep fluid. *J. Appl. Mech. Phys.* **2** (1968), 190–194.

[37] V.E. ZAKHAROV, Space-time spectra of sea waves in the weak turbulence approximation. Seminar talk at the 'Workshop on waves in the Ocean', MSRI, Berkeley (1994).

[38] V.E. ZAKHAROV & A.B. SHABAT, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Zh. Eksp. Teor. Fiz* **61** (1971), 118–134 [Russian]. English translation in *Soviet Physics JETP* **34** (1972), 62–69.

Mathematisches Institut A, Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart
poschel@mathematik.uni-stuttgart.de