

**On an estimate by Sergej Kuksin
concerning a partial differential equation
on a torus with variable coefficients**

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Statement of the Result. We consider the first order partial differential equation

$$-i\partial_{\omega}u + \lambda u + b(x)u = f, \quad x \in \mathbb{T}^n, \quad (1)$$

for functions on the torus $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$, where

$$\partial_{\omega} = \sum_{v=1}^n \omega_v \partial_{x_v}, \quad \omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n.$$

We make the following assumptions.

(A) ω is diophantine: there are constants $\alpha > 0$, $\tau > n$ and $l > 0$ such that

$$|k \cdot \omega + \lambda| \geq \frac{\alpha l}{|k|^{\tau}},$$
$$|k \cdot \omega| \geq \frac{\alpha}{|k|^{\tau}},$$

for all $0 \neq k \in \mathbb{Z}^n$. Also, $|\lambda| \geq \alpha l$.

(B) b is analytic on some complex strip

$$U(s) = \{x : |\operatorname{Im} x| < s\} \subset \mathbb{C}^n$$

around \mathbb{T}^n with mean value zero: $[b] = \int_{\mathbb{T}^n} b(x) dx = 0$, and

$$\|b\|_{s,\tau} \stackrel{\text{def}}{=} \sum_k |\hat{b}_k| |k|^\tau e^{|k|s} \leq \gamma\alpha$$

with some $\gamma > 0$. We also assume $s \leq 1$ for simplicity.

(C) f is analytic on $U(s)$ and bounded:

$$\|f\|_s \stackrel{\text{def}}{=} \sum_k |\hat{f}_k| e^{|k|s} < \infty.$$

Proposition (Kuksin [1]) *Under assumptions (A), (B), (C) equation (1) has a unique solution $u = Lf$ that is analytic on $U(s)$ and satisfies*

$$\|u\|_{s-\sigma} \leq \frac{c}{\alpha l} \|f\|_s \cdot \frac{e^{2\gamma}}{\sigma^\tau}, \quad 0 < \sigma \leq s. \quad (2)$$

Moreover, if

$$\frac{\gamma}{\lambda} \leq \frac{\sigma}{5|\omega|},$$

then also

$$\|u\|_{s-\sigma} \leq \frac{c}{\alpha l} \|f\|_s \cdot \frac{1}{\sigma^{\tau+n+3}}, \quad 0 < \sigma \leq s. \quad (3)$$

The constant c depends only on τ in the first case, and only on τ and n in the second case.

The second estimate – which is the difficult one – is not needed if there is a uniform bound on γ . But if a whole family of equations with no such bound is considered, then it gives a uniform bound for $\|u\|_{s-\sigma}$ provided λ is substantially larger than γ .

Corollary *If*

$$\lambda \geq |\omega| \gamma^{1+\beta}$$

with some $\beta > 0$, then $u = Lf$ satisfies the estimate

$$\|u\|_{s-\sigma} \leq \frac{c}{\alpha l} \|f\|_s \cdot \frac{e^{2(5/\sigma)^{1/\beta}}}{\sigma^{\tau+n+3}}, \quad 0 < \sigma \leq s,$$

where c is a constant which depends only on n and τ .

Indeed, with this assumption we have

$$\frac{\lambda}{\gamma} \geq |\omega| \gamma^\beta.$$

Hence, the second estimate (3) applies, if

$$\gamma^\beta \geq \frac{5}{\sigma}.$$

Otherwise, the first estimate (2) applies, with $\gamma \leq (5/\sigma)^{1/\beta}$. Combinig these two estimates we obtain the estimate of the Corollary.

The proof of the Proposition has two parts. First, the solution $u = Lf$ is constructed by converting (1) via an integrating factor into an equation with constant coefficients. This also provides the first estimate (2). Second, for γ/λ sufficiently small, one considers a sequence of equations (1) where the frequency vector ω is approximated by rational ones and the variable coefficient b by trigonometric polynomials. Each of these equations has a unique periodic solution, which can be represented by an oscillatory integral with a complex valued phase function. By a contour deformation using Cauchy's theorem, these oscillatory integrals are estimated uniformly. Taking limits one obtains the second estimate (3).

This intricate and ingenious scheme was first described by S. Kuksin [1]. We reproduce it here in our style, first to make the result more accessible, and second to apply in [2].

Before giving the details we observe that we may divide equation (1) by α and thus replace ω, λ, b and f by $\alpha^{-1}\omega, \alpha^{-1}\lambda, \alpha^{-1}b$ and $\alpha^{-1}f$, respectively. In the assumptions (A), (B) and (C), α is then replaced by 1. Henceforth, we can assume the normalization

$$\alpha = 1$$

for the rest of our considerations.

The proof of the Proposition now takes N steps, $N = 7$.

1. *Existence and Uniqueness.* To obtain an integrating factor B in (1), solve

$$\partial_\omega B = b, \quad [B] = 0.$$

Then

$$\begin{aligned} \|B\|_s &= \sum_{k \neq 0} |\hat{B}_k| e^{|k|s} = \sum_{k \neq 0} \frac{|\hat{b}_k|}{|k \cdot \omega|} e^{|k|s} \\ &\leq \sum_{k \neq 0} |\hat{b}_k| |k|^\tau e^{|k|s} \leq \|b\|_{s,\tau} \leq \gamma. \end{aligned}$$

Hence, B is analytic in $U(s)$. Now set $u = e^{-iB}v$, $f = e^{-iB}g$ to obtain

$$-i\partial_\omega v + \lambda v = g.$$

This equation has a unique solution v by comparing Fourier coefficients:

$$(k \cdot \omega + \lambda)\hat{v}_k = \hat{g}_k, \quad k \in \mathbb{Z}^n.$$

We get

$$\begin{aligned} \|v\|_{s-\sigma} &= \sum_k |\hat{v}_k| e^{|k|(s-\sigma)} \\ &= \sum_k \frac{|\hat{g}_k|}{|k \cdot \omega + \lambda|} e^{|k|(s-\sigma)} \\ &\leq \sum_k \frac{1}{l} |\hat{g}_k| (1 + |k|^\tau) e^{|k|(s-\sigma)} \\ &\leq \frac{1}{l} \|g\|_s \sup_{t \geq 0} (1 + t^\tau) e^{-\sigma t} \\ &\leq \frac{1}{l} \|g\|_s \frac{c}{\sigma^\tau} \end{aligned}$$

with some constant c depending only on τ .

Going back we obtain the unique analytic solution $u = e^{-iB}v$ of (1) with the estimates

$$\begin{aligned} \|u\|_{s-\sigma} &\leq e^{\|B\|_s} \|v\|_{s-\sigma} \leq e^\gamma \|v\|_{s-\sigma}, \\ \|g\|_s &\leq e^{\|B\|_s} \|f\|_s \leq e^\gamma \|f\|_s, \end{aligned}$$

where we made use of the fact that the norm is multiplicative: $\|uv\|_s \leq \|u\|_s \|v\|_s$. Hence,

$$\|u\|_{s-\sigma} \leq \frac{c}{l} \|f\|_s \frac{e^{2\gamma}}{\sigma^\tau}$$

as stated in (2).

2. *Approximation.* To obtain estimate (3) we now assume that

$$\frac{\gamma}{\lambda} \leq \frac{\sigma}{5|\omega|}, \quad (4)$$

and approximate ω by rational frequency vectors ω_ν . The following is proven at the end; recall that $\alpha = 1$.

Approximation Lemma *There exists a sequence of frequency vectors*

$$\omega_\nu = \frac{2\pi}{T_\nu} \cdot m_\nu$$

with $T_\nu \rightarrow \infty$, $m_\nu \in \mathbb{Z}^n$, and a sequence $K_\nu \rightarrow \infty$ such that for all $\nu \geq 0$,

$$\begin{aligned} \text{(a)} \quad & |\omega - \omega_\nu| \leq \frac{2\pi}{T_\nu}, \\ \text{(b)} \quad & |k \cdot \omega_\nu| \geq \frac{1}{2|k|^\tau}, \quad 0 < |k| < K_\nu, \\ \text{(c)} \quad & |k \cdot \omega_\nu + \lambda| \geq \frac{l}{2|k|^{\tau+n+2}}, \quad |k| \neq 0, \end{aligned}$$

Consequently, also

$$\text{(d)} \quad \frac{\gamma}{\lambda} \leq \frac{\sigma}{4|\omega_\nu|}$$

for large ν .

3. *The periodic problem.* For each large ν we now consider the approximate, periodic problem

$$-i\partial_{\omega_\nu} u + \lambda u + b_\nu(x)u = f,$$

where

$$\omega_\nu = \frac{2\pi}{T_\nu} \cdot m_\nu, \quad b_\nu = \sum_{0 < |k| < K_\nu} \hat{b}_k e^{ik \cdot x}.$$

Fixing ν we drop it from the notation in the following, which hopefully does not lead to confusions.

We may solve again $\partial_\omega B = b$ with $[B] = 0$. Using (b) of the Approximation Lemma we get, as before,

$$\|B\|_s \leq 2 \|b\|_{s,\tau} \leq 2\gamma,$$

and with $u = e^{-iB}v$, $f = e^{-iB}g$, we obtain the equation

$$-i\partial_\omega v + \lambda v = g. \quad (5)$$

Its solution has the integral representation

$$v(x) = \eta_T \int_0^T e^{i\lambda t} g(x + t\omega) dt, \quad \eta_T = \frac{i}{e^{i\lambda T} - 1}. \quad (6)$$

To verify this write $w(t) = v(x + t\omega)$ and $h(t) = g(x + t\omega)$. Both functions are periodic in t with period T . Substituting into (5) we find

$$\frac{d}{dt}w(t) + i\lambda w(t) = ih(t).$$

With an integrating factor $e^{i\lambda t}$ and a subsequent integration over $[0, T]$ this leads to (6).

From this and the substitutions results the representation of u as an oscillatory integral:

$$\begin{aligned} u(x) &= \eta_T \int_0^T e^{i\lambda t + iB(x+t\omega) - iB(x)} f(x + t\omega) dt \\ &= \eta_T \int_0^T e^{i\lambda(t + \delta[B(x+t\omega) - B(x)])} f(x + t\omega) dt \end{aligned} \quad (7)$$

with

$$\delta = \frac{1}{\lambda}.$$

Our aim is to obtain a *uniform* estimate for $x \in U(s - 2\sigma)$. In the following we therefore fix x with $|\operatorname{Im} x| < s - 2\sigma$ and seek such an estimate for $|u(x)|$.

4. *Stationary Phase Transformation.* We want to find a change of the integration variable $t = \phi(r)$ with $\phi(0) = 0$ and $\phi(T) = T$ such that

$$\phi(r) + \delta(B(x + \phi(r)\omega) - B(x)) = r. \quad (8)$$

We then obtain

$$u(x) = \eta_T \int_0^T e^{i\lambda r} f(x + \phi(r)\omega) \phi'(r) dr,$$

which we will use to estimate u .

To find ϕ we lift the problem to the torus \mathbb{T}^n and write

$$\phi(r) = r + \psi(r\omega), \quad \psi: \mathbb{T}^n \rightarrow \mathbb{C}.$$

Also, let $r\omega = \xi \in \mathbb{T}^n$. From (8) we obtain the equation

$$\psi(\xi) + \delta(B(x + \xi + \psi(\xi)\omega) - B(x)) = 0 \quad (9)$$

which we solve by a contraction argument.

Consider the space \mathfrak{A} of functions $\psi: U(\sigma) \rightarrow \mathbb{C}$ with

- (i) ψ analytic and periodic on $U(\sigma)$,
- (ii) $\psi(0) = 0$,
- (iii) $|\psi|_\sigma = \sup_{\xi \in U(\sigma)} |\psi(\xi)| \leq 4\gamma\delta$.

Define a map \mathcal{T} on \mathfrak{A} by

$$\mathcal{T}(\psi)(\xi) = -\delta(B(x + \xi + \psi(\xi)\omega) - B(x))$$

To show that \mathcal{T} is well defined we use that

$$\gamma\delta = \frac{\gamma}{\lambda} \leq \frac{\sigma}{4|\omega|}$$

by (d) in the Approximation Lemma. Then

$$|\operatorname{Im}(\psi(\xi)\omega)| \leq |\psi|_\sigma |\omega| \leq 4\gamma\delta |\omega| \leq \sigma,$$

hence $|\operatorname{Im}(x + \xi + \psi(\xi)\omega)| < s - 2\sigma + \sigma + \sigma = s$. It follows that $B(x + \xi + \psi(\xi)\omega)$ and thus $\mathcal{T}(\psi)$ are well defined.

\mathcal{T} maps \mathfrak{A} into \mathfrak{A} : (i) and (ii) clearly hold, and, as $|B|_\sigma \leq \|B\|_s \leq 2\gamma$,

$$|\mathcal{T}(\psi)|_\sigma \leq 2\delta |B|_\sigma \leq 4\gamma\delta.$$

Further, \mathcal{T} is a contraction:

$$\begin{aligned} \mathcal{T}(\psi)(\xi) - \mathcal{T}(\chi)(\xi) &= \delta(B(x + \xi + \chi\omega) - B(x + \xi + \psi\omega)) \\ &= \delta \int_0^1 \langle dB(x + \xi(s)), (\psi - \chi)\omega \rangle ds \\ &= \delta \int_0^1 b(x + \xi(s))(\psi - \chi) ds \end{aligned}$$

with $\xi(s) = \xi + ((1-s)\chi + s\psi)\omega$, hence, using assumption (B),

$$|\mathcal{T}(\psi) - \mathcal{T}(\chi)|_\sigma \leq \delta \|b\|_{s,\tau} |\psi - \chi|_\sigma \leq \gamma\delta |\psi - \chi|_\sigma \leq \frac{1}{2} |\psi - \chi|_\sigma,$$

since $\gamma\delta \leq \sigma/4 |\omega| \leq 1/4 |\omega|$, and $|\omega| \geq \frac{1}{2}$ for all approximating frequencies.

Thus there exists a unique analytic solution $\psi \in \mathfrak{A}$ of equation (9). Consequently, ϕ with $\phi(r) = r + \psi(r\omega)$ solves equation (8). In particular, $\phi - \operatorname{id}$ has period T . Moreover, from the fixed point equation (9) one gets

$$\partial_\omega \psi = -\delta \langle dB, \omega + (\partial_\omega \psi)\omega \rangle = -\delta b(1 + \partial_\omega \psi)$$

and, again using assumption (B),

$$|\partial_\omega \psi|_\sigma \leq \gamma\delta(1 + |\partial_\omega \psi|_\sigma) \leq \frac{1}{2}(1 + |\partial_\omega \psi|_\sigma),$$

hence $|\partial_\omega \psi|_\sigma \leq 1$.

5. Transformation and Estimate. We observe that ϕ maps the interval $[0, T]$ into a curve Γ in $U(\sigma)$ with the *same* endpoints as $[0, T]$. By Cauchy's theorem, the integral in (7) over $[0, T]$ is the same as over Γ . Thus, substituting $t = \phi(r) = r + \psi(r\omega)$ in the integral (7) we obtain

$$u(x) = \eta_T \int_0^T e^{i\lambda r} f(x + \phi(r)\omega) \phi'(r) dr = \eta_T \int_0^T e^{i\lambda r} h(x; \xi) | \xi = r\omega dr$$

with

$$h(x; \xi) = f(x + \xi + \psi(\xi)\omega)(1 + \partial_\omega \psi(\xi))$$

analytic on $U(\sigma)$, and

$$|h(x; \cdot)|_\sigma \leq 2|f|_s.$$

The above integral for $u(x)$ is now easily evaluated. Since h is periodic in ξ ,

$$h(x; \xi) = \sum_k \hat{h}_k(x) e^{ik \cdot \xi}, \quad |\hat{h}_k| \leq |h|_\sigma e^{-|k|\sigma}.$$

Moreover

$$\int_0^T e^{i\lambda r} e^{ik \cdot \xi} |_\xi = r\omega dr = \int_0^T e^{i(\lambda+k \cdot \omega)r} dr = -i \frac{e^{i\lambda T} - 1}{k \cdot \omega + \lambda} = \frac{\eta_T^{-1}}{k \cdot \omega + \lambda}.$$

Hence we obtain

$$u(x) = \sum_k \frac{\hat{h}_k(x)}{k \cdot \omega + \lambda}.$$

With (c) of the Approximation Lemma, one obtains for every $x \in U(s - 2\sigma)$ the estimate

$$|u(x)| \leq \frac{2}{l} \sum_{k \neq 0} |k|^{\tau+n+2} e^{-|k|\sigma} |h|_\sigma + \frac{1}{\lambda} |h|_\sigma \leq \frac{1}{l} |f|_s \cdot \frac{c}{\sigma^{\tau+n+3}},$$

where c depends only on n and τ . We used that $\lambda \geq l$ by hypotheses (A) for $\alpha = 1$.

6. Taking Limits. Summarizing the preceding arguments we obtain the following. For every frequency vector

$$\omega_\nu = \frac{2\pi}{T_\nu} \cdot m_\nu, \quad m_\nu \in \mathbb{Z}^n,$$

satisfying (b), (c) and (d) the equation $-i\partial_{\omega_\nu} u + \lambda u + b_\nu u = f$ has a unique analytic solution u_ν satisfying

$$|u_\nu|_{s-2\sigma} \leq \frac{1}{l} |f|_s \cdot \frac{c}{\sigma^{\tau+n+3}}, \quad 0 < \sigma \leq s.$$

Letting $\nu \rightarrow \infty$ and taking (a) of the Approximation Lemma into account, we obtain approximating sequences $\omega_\nu \rightarrow \omega$, $b_\nu \rightarrow b$. The corresponding u_ν are uniformly

bounded. Hence we can choose a convergent subsequence converging to some solution u of $-i\partial_\omega u + \lambda u + bu = f$ with

$$|u|_{s-2\sigma} \leq \frac{1}{l} |f|_s \cdot \frac{c}{\sigma^{\tau+n+3}}.$$

From this the second estimate (3) of the proposition follows. Incidentally, since the solution u is unique, actually the whole sequence u_ν converges to u .

7. Proof of the Approximation Lemma. Suppose ω satisfies assumption (A). For each t -interval

$$I = [T, T + \Delta], \quad T \geq \Delta = \frac{2\pi}{|\omega|}$$

there is an integer vector $m \in \mathbb{Z}^n$ such that

$$|t\omega - 2\pi m| \leq 2\pi, \quad t \in I.$$

Hence,

$$|\omega - \omega_t| \leq \frac{2\pi}{T}, \quad \omega_t = \frac{2\pi}{t} \cdot m,$$

for each $t \in I$. This gives (a).

To obtain (b) we estimate, for $t \in I$ arbitrary,

$$|k \cdot \omega_t| \geq |k \cdot \omega| - |k \cdot (\omega - \omega_t)| \geq \frac{1}{|k|^\tau} - |k| \frac{2\pi}{T} \geq \frac{1}{2|k|^\tau}$$

for $4\pi |k|^{\tau+1} \leq T$, that is, for

$$|k| \leq K_T \stackrel{\text{def}}{=} \left(\frac{T}{4\pi} \right)^{1/(\tau+1)}.$$

This gives (b).

For (c) we estimate similarly

$$|k \cdot \omega_t + \lambda| \geq |k \cdot \omega + \lambda| - |k \cdot (\omega - \omega_t)| \geq \frac{l}{|k|^\tau} - |k| \frac{2\pi}{T} \geq \frac{l}{2|k|^\tau}$$

for $t \in I$ and $4\pi |k|^{\tau+1} \leq lT$. So it remains to consider the case

$$4\pi |k|^{\tau+1} \geq lT. \tag{10}$$

Assume that for some $t \in I$,

$$|k \cdot \omega_t + \lambda| \leq \frac{l}{2},$$

otherwise there is nothing to do. In particular, we have $|k \cdot \omega_t| \geq l/2$, since $\lambda \geq l$. As long as this holds, we have for $\varphi(t) \stackrel{\text{def}}{=} k \cdot \omega_t + \lambda$ the estimate

$$|\varphi'(t)| = \frac{1}{t} |k \cdot \omega_t| \geq \frac{l}{2t} \geq \frac{l}{4T}.$$

Hence the measure of the subset of the t -interval I where

$$|k \cdot \omega_t + \lambda| \leq \frac{l}{2|k|^{\tau+n+2}} \tag{11}$$

can be estimated by

$$\frac{4T}{l} \frac{l}{2|k|^{\tau+n+2}} = \frac{2T}{|k|^{\tau+n+2}}.$$

Summing over all $k \in \mathbb{Z}^n$ with (10) the total measure of the subset of t -values in I satisfying (11) can be estimated by

$$c(n) \cdot \frac{T}{K_0^{\tau+2}},$$

where $c(n)$ depends only on n , and K_0 is the lower bound of $|k|$ in (10). That is,

$$K_0 = c(l)T^{1/(\tau+1)}.$$

Hence the measure is bounded by

$$c(n, l)T^{-1/(\tau+1)} < \frac{2\pi}{|\omega|} = |I|,$$

if T is sufficiently large. Note that the dependence of the constant on l is irrelevant here, since we want to make T large anyhow, after l is given. Hence for each large T we can find $t \in [T, T + \Delta]$ so that (11) does *not* hold. That is, for such t , (c) holds. This proves the lemma.

References

- [1] S.B. KUKSIN, A KAM theorem for equations of the Korteweg-de Vries type. Preprint (1995).
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