First Version Jan 31, 1996
Updated Oct 8, 1996

On an estimate by Sergej Kuksin concerning a partial differential equation on a torus with variable coefficients

THOMAS KAPPELER — *Universität Zürich*
JÜRGEN PÖSCHEL — *Universität Stuttgart*

Statement of the Result. We consider the first order partial differential equation

$$-\mathrm{i}\partial_\omega u + \lambda u + b(x)u = f, \qquad x \in \mathbb{T}^n, \tag{1}$$

for functions on the torus $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$, where

$$\partial_\omega = \sum_{\nu=1}^{n} \omega_\nu \partial_{x_\nu}, \qquad \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n.$$

We make the following assumptions.

(A) ω is diophantine: there are constants $\alpha > 0$, $\tau > n$ and $l > 0$ such that

$$|k \cdot \omega + \lambda| \geq \frac{\alpha l}{|k|^\tau},$$
$$|k \cdot \omega| \geq \frac{\alpha}{|k|^\tau},$$

for all $0 \neq k \in \mathbb{Z}^n$. Also, $|\lambda| \geq \alpha l$.

(B) b is analytic on some complex strip

$$U(s) = \{x : |\mathrm{Im}\, x| < s\} \subset \mathbb{C}^n$$

around $\mathbb{T}^n$ with mean value zero: $[b] = \int_{\mathbb{T}^n} b(x)\, dx = 0$, and

$$\|b\|_{s,\tau} \stackrel{\mathrm{def}}{=} \sum_k |\hat{b}_k|\, |k|^\tau\, e^{|k|s} \le \gamma\alpha$$

with some $\gamma > 0$. We also assume $s \le 1$ for simplicity.

(C) f is analytic on $U(s)$ and bounded:

$$\|f\|_s \stackrel{\mathrm{def}}{=} \sum_k |\hat{f}_k| e^{|k|s} < \infty.$$

Proposition (Kuksin [1]) *Under assumptions* (A), (B), (C) *equation* (1) *has a unique solution $u = Lf$ that is analytic on $U(s)$ and satisfies*

$$\|u\|_{s-\sigma} \le \frac{c}{\alpha l}\, \|f\|_s \cdot \frac{e^{2\gamma}}{\sigma^\tau}, \qquad 0 < \sigma \le s. \tag{2}$$

Moreover, if

$$\frac{\gamma}{\lambda} \le \frac{\sigma}{5\,|\omega|},$$

then also

$$\|u\|_{s-\sigma} \le \frac{c}{\alpha l}\, \|f\|_s \cdot \frac{1}{\sigma^{\tau+n+3}}, \qquad 0 < \sigma \le s. \tag{3}$$

The constant c depends only on τ in the first case, and only on τ and n in the second case.

The second estimate – which is the difficult one – is not needed if there is a uniform bound on γ. But if a whole family of equations with no such bound is considered, then it gives a uniform bound for $\|u\|_{s-\sigma}$ provided λ is substantially larger than γ.

Corollary *If*

$$\lambda \ge |\omega|\, \gamma^{1+\beta}$$

with some $\beta > 0$, then $u = Lf$ satisfies the estimate

$$\|u\|_{s-\sigma} \le \frac{c}{\alpha l}\, \|f\|_s \cdot \frac{e^{2(5/\sigma)^{1/\beta}}}{\sigma^{\tau+n+3}}, \qquad 0 < \sigma \le s,$$

where c is a constant which depends only on n and τ.

Indeed, with this assumption we have

$$\frac{\lambda}{\gamma} \ge |\omega|\, \gamma^\beta.$$

Hence, the second estimate (3) applies, if

$$\gamma^\beta \ge \frac{5}{\sigma}.$$

Otherwise, the first estimate (2) applies, with $\gamma \le (5/\sigma)^{1/\beta}$. Combinig these two estimates we obtain the estimate of the Corollary.

The proof of the Proposition has two parts. First, the solution $u = Lf$ is constructed by converting (1) via an integrating factor into an equation with constant coefficients. This also provides the first estimate (2). Second, for γ/λ sufficiently small, one considers a sequence of equations (1) where the frequency vector ω is approximated by rational ones and the variable coefficient b by trigonometric polynomials. Each of these equations has a unique periodic solution, which can be represented by an oscillatory integral with a complex valued phase function. By a contour deformation using Cauchy's theorem, these oscillatory integrals are estimated uniformly. Taking limits one obtains the second estimate (3).

This intricate and ingenious scheme was first described by S. Kuksin [1]. We reproduce it here in our style, first to make the result more accessible, and second to apply in [2].

Before giving the details we observe that we may divide equation (1) by α and thus replace ω, λ, b and f by $\alpha^{-1}\omega, \alpha^{-1}\lambda, \alpha^{-1}b$ and $\alpha^{-1}f$, respectively. In the assumptions (A), (B) and (C), α is then replaced by 1. Henceforth, we can assume the normalization

$$\alpha = 1$$

for the rest of our considerations.

The proof of the Proposition now takes N steps, $N = 7$.

1. Existence and Uniqueness. To obtain an integrating factor B in (1), solve

$$\partial_\omega B = b, \quad [B] = 0.$$

Then

$$\begin{aligned}\|B\|_s &= \sum_{k\neq 0} |\hat{B}_k| e^{|k|s} = \sum_{k\neq 0} \frac{|\hat{b}_k|}{|k\cdot\omega|} e^{|k|s} \\ &\le \sum_{k\neq 0} |\hat{b}_k|\,|k|^\tau\, e^{|k|s} \le \|b\|_{s,\tau} \le \gamma.\end{aligned}$$

Hence, B is analytic in $U(s)$. Now set $u = e^{-\mathrm{i}B} v,\ f = e^{-\mathrm{i}B} g$ to obtain

$$-\mathrm{i}\partial_\omega v + \lambda v = g.$$

This equation has a unique solution v by comparing Fourier coefficients:

$$(k\cdot\omega + \lambda)\hat{v}_k = \hat{g}_k, \qquad k \in \mathbb{Z}^n.$$

We get

$$\begin{aligned}\|v\|_{s-\sigma} &= \sum_k |\hat{v}_k| e^{|k|(s-\sigma)} \\ &= \sum_k \frac{|\hat{g}_k|}{|k\cdot\omega+\lambda|} e^{|k|(s-\sigma)} \\ &\le \sum_k \frac{1}{l} |\hat{g}_k| (1+|k|^\tau) e^{|k|(s-\sigma)} \\ &\le \frac{1}{l}\, \|g\|_s \sup_{t\ge 0} (1+t^\tau) e^{-\sigma t} \\ &\le \frac{1}{l}\, \|g\|_s\, \frac{c}{\sigma^\tau}\end{aligned}$$

with some constant c depending only on τ.

Going back we obtain the unique analytic solution $u = e^{-\mathrm{i}B} v$ of (1) with the estimates

$$\begin{aligned}\|u\|_{s-\sigma} &\le e^{\|B\|_s}\, \|v\|_{s-\sigma} \le e^\gamma\, \|v\|_{s-\sigma}\,, \\ \|g\|_s &\le e^{\|B\|_s}\, \|f\|_s \quad\ \le e^\gamma\, \|f\|_s\,,\end{aligned}$$

where we made use of the fact that the norm is multiplicative: $\|uv\|_s \le \|u\|_s\, \|v\|_s$. Hence,

$$\|u\|_{s-\sigma} \le \frac{c}{l}\, \|f\|_s\, \frac{e^{2\gamma}}{\sigma^\tau}$$

as stated in (2).

2. Approximation. To obtain estimate (3) we now assume that

$$\frac{\gamma}{\lambda} \le \frac{\sigma}{5\,|\omega|}, \tag{4}$$

and approximate ω by rational frequency vectors ω_ν. The following is proven at the end; recall that $\alpha = 1$.

Approximation Lemma *There exists a sequence of frequency vectors*

$$\omega_\nu = \frac{2\pi}{T_\nu}\cdot m_\nu$$

with $T_\nu \to \infty$, $m_\nu \in \mathbb{Z}^n$, and a sequence $K_\nu \to \infty$ such that for all $\nu \ge 0$,

(a) $$|\omega - \omega_\nu| \le \frac{2\pi}{T_\nu},$$

(b) $$|k\cdot\omega_\nu| \ge \frac{1}{2\,|k|^\tau}, \qquad 0 < |k| < K_\nu,$$

(c) $$|k\cdot\omega_\nu + \lambda| \ge \frac{l}{2\,|k|^{\tau+n+2}}, \qquad |k| \ne 0,$$

Consequently, also

(d) $$\frac{\gamma}{\lambda} \le \frac{\sigma}{4\,|\omega_\nu|}$$

for large ν.

3. The periodic problem. For each large ν we now consider the approximate, periodic problem

$$-\mathrm{i}\partial_{\omega_\nu} u + \lambda u + b_\nu(x) u = f,$$

where

$$\omega_\nu = \frac{2\pi}{T_\nu} \cdot m_\nu, \qquad b_\nu = \sum_{0<|k|<K_\nu} \hat{b}_k e^{\mathrm{i}k\cdot x}.$$

Fixing ν we drop it from the notation in the following, which hopefully does not lead to confusions.

We may solve again $\partial_\omega B = b$ with $[B] = 0$. Using (b) of the Approximation Lemma we get, as before,

$$\|B\|_s \le 2\,\|b\|_{s,\tau} \le 2\gamma,$$

and with $u = e^{-\mathrm{i}B}v,\ f = e^{-\mathrm{i}B}g$, we obtain the equation

$$-\mathrm{i}\partial_\omega v + \lambda v = g. \tag{5}$$

Its solution has the integral representation

$$v(x) = \eta_T \int_0^T e^{\mathrm{i}\lambda t} g(x + t\omega)\, dt, \qquad \eta_T = \frac{\mathrm{i}}{e^{\mathrm{i}\lambda T} - 1}. \tag{6}$$

To verify this write $w(t) = v(x + t\omega)$ and $h(t) = g(x + t\omega)$. Both functions are periodic in t with period T. Substituting into (5) we find

$$\frac{d}{dt} w(t) + \mathrm{i}\lambda w(t) = \mathrm{i}h(t).$$

With an integrating factor $e^{\mathrm{i}\lambda t}$ and a subsequent integration over $[0, T]$ this leads to (6).

From this and the substitutions results the representation of u as an oscillatory integral:

$$\begin{aligned} u(x) &= \eta_T \int_0^T e^{\mathrm{i}\lambda t + \mathrm{i}B(x+t\omega) - \mathrm{i}B(x)} f(x + t\omega)\, dt \\ &= \eta_T \int_0^T e^{\mathrm{i}\lambda(t + \delta[B(x+t\omega) - B(x)])} f(x + t\omega)\, dt \end{aligned} \tag{7}$$

with

$$\delta = \frac{1}{\lambda}.$$

Our aim is to obtain a *uniform* estimate for $x \in U(s - 2\sigma)$. In the following we therefore *fix* x with $|\mathrm{Im}\, x| < s - 2\sigma$ and seek such an estimate for $|u(x)|$.

4. Stationary Phase Transformation. We want to find a change of the integration variable $t = \phi(r)$ with $\phi(0) = 0$ and $\phi(T) = T$ such that

$$\phi(r) + \delta\big(B(x + \phi(r)\omega) - B(x)\big) = r. \tag{8}$$

We then obtain

$$u(x) = \eta_T \int_0^T e^{\mathrm{i}\lambda r} f(x + \phi(r)\omega)\phi'(r)\, dr,$$

which we will use to estimate u.

To find ϕ we lift the problem to the torus $\mathbb{T}^n$ and write

$$\phi(r) = r + \psi(r\omega), \qquad \psi : \mathbb{T}^n \to \mathbb{C}.$$

Also, let $r\omega = \xi \in \mathbb{T}^n$. From (8) we obtain the equation

$$\psi(\xi) + \delta\big(B(x + \xi + \psi(\xi)\omega) - B(x)\big) = 0 \tag{9}$$

which we solve by a contraction argument.

Consider the space $\mathfrak{A}$ of functions $\psi : U(\sigma) \to \mathbb{C}$ with

(i) ψ analytic and periodic on $U(\sigma)$,

(ii) $\psi(0) = 0$,

(iii) $|\psi|_\sigma = \sup_{\xi \in U(\sigma)} |\psi(\xi)| \le 4\gamma\delta$.

Define a map $\mathcal{T}$ on $\mathfrak{A}$ by

$$\mathcal{T}(\psi)(\xi) = -\delta\big(B(x + \xi + \psi(\xi)\omega) - B(x)\big)$$

To show that $\mathcal{T}$ is well defined we use that

$$\gamma\delta = \frac{\gamma}{\lambda} \le \frac{\sigma}{4\,|\omega|}$$

by (d) in the Approximation Lemma. Then

$$|\mathrm{Im}(\psi(\xi)\omega)| \le |\psi|_\sigma\, |\omega| \le 4\gamma\delta\, |\omega| \le \sigma,$$

hence $|\mathrm{Im}(x+\xi+\psi(\xi)\omega)| < s-2\sigma+\sigma+\sigma = s$. It follows that $B(x+\xi+\psi(\xi)\omega)$ and thus $\mathcal{T}(\psi)$ are well defined.

$\mathcal{T}$ maps $\mathfrak{A}$ into $\mathfrak{A}$: (i) and (ii) clearly hold, and, as $|B|_\sigma \le \|B\|_s \le 2\gamma$,

$$|\mathcal{T}(\psi)|_\sigma \le 2\delta\,|B|_\sigma \le 4\gamma\delta.$$

Further, $\mathcal{T}$ is a contraction:

$$\begin{aligned}\mathcal{T}(\psi)(\xi) - \mathcal{T}(\chi)(\xi) &= \delta\big(B(x+\xi+\chi\omega) - B(x+\xi+\psi\omega)\big)\\ &= \delta\int_0^1 \langle dB(x+\xi(s)), (\psi-\chi)\omega\rangle\, ds\\ &= \delta\int_0^1 b(x+\xi(s))(\psi-\chi)\, ds\end{aligned}$$

with $\xi(s) = \xi + ((1-s)\chi + s\psi)\omega$, hence, using assumption (B),

$$|\mathcal{T}(\psi) - \mathcal{T}(\chi)|_\sigma \le \delta\,\|b\|_{s,\tau}\,|\psi-\chi|_\sigma \le \gamma\delta\,|\psi-\chi|_\sigma \le \frac{1}{2}\,|\psi-\chi|_\sigma\,,$$

since $\gamma\delta \le \sigma/4\,|\omega| \le 1/4\,|\omega|$, and $|\omega| \ge \frac{1}{2}$ for all approximating frequencies.

Thus there exists a unique analytic solution $\psi \in \mathfrak{A}$ of equation (9). Consequently, ϕ with $\phi(r) = r + \psi(r\omega)$ solves equation (8). In particular, $\phi - \mathrm{id}$ has period T. Moreover, from the fixed point equation (9) one gets

$$\partial_\omega\psi = -\delta\,\langle dB, \omega + (\partial_\omega\psi)\omega\rangle = -\delta b(1+\partial_\omega\psi)$$

and, again using assumption (B),

$$|\partial_\omega\psi|_\sigma \le \gamma\delta(1+|\partial_\omega\psi|_\sigma) \le \frac{1}{2}(1+|\partial_\omega\psi|_\sigma),$$

hence $|\partial_\omega\psi|_\sigma \le 1$.

5. Transformation and Estimate. We observe that ϕ maps the interval $[0,T]$ into a curve Γ in $U(\sigma)$ with the *same* endpoints as $[0,T]$. By Cauchy's theorem, the integral in (7) over $[0,T]$ is the same as over Γ. Thus, substituting $t=\phi(r) = r+\psi(r\omega)$ in the integral (7) we obtain

$$u(x) = \eta_T\int_0^T e^{\mathrm{i}\lambda r} f(x+\phi(r)\omega)\phi'(r)\,dr = \eta_T\int_0^T e^{\mathrm{i}\lambda r} h(x;\xi)\big|\xi = r\omega dr$$

with

$$h(x;\xi) = f(x+\xi+\psi(\xi)\omega)(1+\partial_\omega\psi(\xi))$$

analytic on $U(\sigma)$, and

$$|h(x;\,\cdot)|_\sigma \le 2\,|f|_s\,.$$

The above integral for $u(x)$ is now easily evaluated. Since h is periodic in ξ,

$$h(x;\xi) = \sum_k \hat h_k(x) e^{\mathrm{i}k\cdot\xi}, \qquad |\hat h_k| \le |h|_\sigma\, e^{-|k|\sigma}.$$

Moreover

$$\int_0^T e^{\mathrm{i}\lambda r} e^{\mathrm{i}k\cdot\xi}\big|\xi = r\omega dr = \int_0^T e^{\mathrm{i}(\lambda+k\cdot\omega)r}\,dr = -\mathrm{i}\frac{e^{\mathrm{i}\lambda T}-1}{k\cdot\omega+\lambda} = \frac{\eta_T^{-1}}{k\cdot\omega+\lambda}.$$

Hence we obtain

$$u(x) = \sum_k \frac{\hat h_k(x)}{k\cdot\omega+\lambda}.$$

With (c) of the Approximation Lemma, one obtains for every $x \in U(s-2\sigma)$ the estimate

$$|u(x)| \le \frac{2}{l}\sum_{k\ne 0}|k|^{\tau+n+2}\,e^{-|k|\sigma}\,|h|_\sigma + \frac{1}{\lambda}\,|h|_\sigma \le \frac{1}{l}\,|f|_s\cdot\frac{c}{\sigma^{\tau+n+3}},$$

where c depends only on n and τ. We used that $\lambda \ge l$ by hypotheses (A) for $\alpha = 1$.

6. Taking Limits. Summarizing the preceding arguments we obtain the following. For every frequency vector

$$\omega_\nu = \frac{2\pi}{T_\nu}\cdot m_\nu, \qquad m_\nu \in \mathbb{Z}^n,$$

satisfying (b), (c) and (d) the equation $-\mathrm{i}\partial_{\omega_\nu}u + \lambda u + b_\nu u = f$ has a unique analytic solution u_ν satisfying

$$|u_\nu|_{s-2\sigma} \le \frac{1}{l}\,|f|_s\cdot\frac{c}{\sigma^{\tau+n+3}}, \qquad 0<\sigma\le s.$$

Letting $\nu\to\infty$ and taking (a) of the Approximation Lemma into account, we obtain approximating sequences $\omega_\nu \to \omega$, $b_\nu \to b$. The corresponding u_ν are uniformly bounded. Hence we can choose a convergent subsequence converging to some solution u of $-\mathrm{i}\partial_\omega u + \lambda u + bu = f$ with

$$|u|_{s-2\sigma} \le \frac{1}{l}\,|f|_s \cdot \frac{c}{\sigma^{\tau+n+3}}.$$

From this the second estimate (3) of the proposition follows. Incidentally, since the solution u is unique, actually the whole sequence u_ν converges to u.

7. Proof of the Approximation Lemma. Suppose ω satisfies assumption (A). For each t-interval

$$I = [T, T+\Delta], \qquad T \ge \Delta = \frac{2\pi}{|\omega|}$$

there is an integer vector $m \in \mathbb{Z}^n$ such that

$$|t\omega - 2\pi m| \le 2\pi, \qquad t \in I.$$

Hence,

$$|\omega - \omega_t| \le \frac{2\pi}{T}, \qquad \omega_t = \frac{2\pi}{t}\cdot m,$$

for each $t \in I$. This gives (a).

To obtain (b) we estimate, for $t \in I$ arbitrary,

$$|k\cdot\omega_t| \ge |k\cdot\omega| - |k\cdot(\omega-\omega_t)| \ge \frac{1}{|k|^\tau} - |k|\,\frac{2\pi}{T} \ge \frac{1}{2\,|k|^\tau}$$

for $4\pi\,|k|^{\tau+1} \le T$, that is, for

$$|k| \le K_T \overset{\text{def}}{=} \left(\frac{T}{4\pi}\right)^{1/(\tau+1)}.$$

This gives (b).

For (c) we estimate similarly

$$|k\cdot\omega_t + \lambda| \ge |k\cdot\omega+\lambda| - |k\cdot(\omega-\omega_t)| \ge \frac{l}{|k|^\tau} - |k|\,\frac{2\pi}{T} \ge \frac{l}{2\,|k|^\tau}$$

for $t \in I$ and $4\pi\,|k|^{\tau+1} \le lT$. So it remains to consider the case

$$4\pi\,|k|^{\tau+1} \ge lT. \tag{10}$$

Assume that for some $t \in I$,

$$|k\cdot\omega_t + \lambda| \le \frac{l}{2},$$

otherwise there is nothing to do. In particular, we have $|k\cdot\omega_t| \ge l/2$, since $\lambda \ge l$. As long as this holds, we have for $\varphi(t) \overset{\text{def}}{=} k\cdot\omega_t + \lambda$ the estimate

$$\left|\varphi'(t)\right| = \frac{1}{t}\,|k\cdot\omega_t| \ge \frac{l}{2t} \ge \frac{l}{4T}.$$

Hence the measure of the subset of the t-interval I where

$$|k\cdot\omega_t + \lambda| \le \frac{l}{2\,|k|^{\tau+n+2}} \tag{11}$$

can be estimated by

$$\frac{4T}{l}\,\frac{l}{2\,|k|^{\tau+n+2}} = \frac{2T}{|k|^{\tau+n+2}}.$$

Summing over all $k \in \mathbb{Z}^n$ with (10) the total measure of the subset of t-values in I satisfying (11) can be estimated by

$$c(n)\cdot\frac{T}{K_0^{\tau+2}},$$

where $c(n)$ depends only on n, and K_0 is the lower bound of $|k|$ in (10). That is,

$$K_0 = c(l)T^{1/(\tau+1)}.$$

Hence the measure is bounded by

$$c(n,l)T^{-1/(\tau+1)} < \frac{2\pi}{|\omega|} = |I|,$$

if T is sufficiently large. Note that the dependence of the constant on l is irrelevant here, since we want to make T large anyhow, after l is given. Hence for each large T we can find $t \in [T, T+\Delta]$ so that (11) does *not* hold. That is, for such t, (c) holds. This proves the lemma.

References

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Universität Zürich, Institut für Reine Mathematik, Winterthurerstrasse 190, CH-8057 Zürich
tk@math.unizh.ch

Mathematisches Institut A, Universität Stuttgart, Pfaffenwaldring 57, D–70569 Stuttgart
poschel@mathematik.uni-stuttgart.de