

A Note on Gaps of Hill's Equation

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1 Results

We consider the differential operator

$$L = -\frac{d^2}{dx^2} + q, \quad q \in L^2 = L^2(S^1, \mathbb{R})$$

on the interval $[0, 1]$ endowed with periodic or anti-periodic boundary conditions:

$$y(0) = y(1), \quad y'(0) = y'(1)$$

or

$$y(0) = -y(1), \quad y'(0) = -y'(1).$$

The corresponding differential equation

$$-y'' + qy = \lambda y$$

is also known as *Hill's equation with potential q* .

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It is well known that the spectrum of L is pure point and consists of an unbounded sequence of *periodic eigenvalues*

$$\lambda_0(q) < \lambda_1(q) \leq \lambda_2(q) < \lambda_3(q) \leq \lambda_4(q) < \dots$$

Equality or inequality may occur in every place with a ‘ \leq ’-sign, and one speaks of the *gaps* $(\lambda_{2n-1}(q), \lambda_{2n}(q))$ of the potential q and its *gap lengths*

$$\gamma_n(q) = \lambda_{2n}(q) - \lambda_{2n-1}(q), \quad n \geq 1.$$

If some gap length is zero, one speaks of a *collapsed gap*, otherwise of an *open gap*.

The purpose of this note is to give new, short proofs of two facts relating these gap lengths to the regularity of the potential q . To formulate these results, denote by $H^m = H^m(S^1)$ the Sobolev space of m times weakly differentiable functions of period 1. That is,

$$H^m = \{u \in L^2(S^1, \mathbb{R}) : \|u\|_m < \infty\},$$

where $\|u\|_m^2 = |\hat{u}(0)|^2 + \sum_{n \neq 0} n^{2m} |\hat{u}(n)|^2$ is defined in terms of the discrete Fourier transform \hat{u} of u .

The following result was first proven by Marčenko & Ostrowski¹ [12], using inverse spectral theory. Their approach was later simplified by Garnett & Trubowitz [3] and generalized in [7]. For a more elementary proof see also [4, 5].

Theorem 1.1 *The gap lengths satisfy*

$$\sum_{n \geq 1} n^{2m} \gamma_n^2 < \infty$$

locally uniformly on H^m for any $m \geq 0$.

In fact, Marčenko & Ostrowski¹ [12] also prove the converse statement: if the gaps of a given potential in H^0 are as above, then this potential is in H^m . For further results in this direction see also [3, 7, 8].

The second result concerns the density of finite gap potentials, which are potentials with only a finite number of open gaps.

Theorem 1.2 *Finite gap potentials are dense in H^m for any $m \geq 0$.*

This result was conjectured by Novikov [14] (see also Lax [9]) and first proven by Marčenko & Ostrovskii [12]. See also [3, 7, 10, 11] and others, for example [13]. While these approaches use inverse spectral theory, our proof uses only asymptotic

properties of some spectral data. In this respect, the first proof sans inverse spectral theory appeared in [1] for the case $m = 0$.

We point out that Theorems 1.1 and 1.2 are used in the proof of the normal form theorem for KdV in [6], asserting that KdV admits Birkhoff coordinates in any Sobolev space H^m , $m \geq 0$. The case $m = 0$ of these two theorems is treated in detail in [6], while the case $m \geq 1$ is quoted from other sources. With this note we supply a proof for the case $m \geq 1$ along the same lines as for $m = 0$ – as it should have appeared in [6] ...

The rest of this note is devoted to new proofs of these two results. Indeed, we also show that Theorem 1.1 holds in some complex neighbourhood of H^m for each $m \geq 0$, and that there is a similar result for quantities involving Dirichlet eigenvalues.

2 Some Background

Denote by y_1, y_2 the fundamental solution of $-y'' + qy = \lambda y$ satisfying

$$\begin{aligned} y_1(0, \lambda, q) &= 1, & y_2(0, \lambda, q) &= 0, \\ y_1'(0, \lambda, q) &= 0, & y_2'(0, \lambda, q) &= 1. \end{aligned}$$

The spectrum of the operator

$$L = -\frac{d^2}{dx^2} + q, \quad q \in L^2,$$

endowed with Dirichlet boundary conditions is called the *Dirichlet spectrum of q* and coincides with the zero set of the entire function $y_2(1, \cdot, q)$. It is an unbounded sequence of *Dirichlet eigenvalues*

$$\mu_1(q) < \mu_2(q) < \mu_3(q) < \dots,$$

which are all simple. With each eigenvalue one can associate a unique *Dirichlet eigenfunction*

$$g_n = \frac{y_2}{\|y_2\|} \Big|_{\lambda=\mu_n}, \quad n \geq 1.$$

Besides the μ_n we also need to consider the quantities

$$\kappa_n(q) = \log(-1)^n y_2'(1, \mu_n(q), q), \quad n \geq 1,$$

which measure the terminal velocities of the eigenfunctions.

The following facts are proven in [15]. We write $\ell_m^2(n)$ for the n -th term of a generic sequence $x = (x_n)_{n \geq 1}$ with

$$\sum_{n \geq 1} n^{2m} |x_n|^2 < \infty,$$

and $\ell^2(n)$ for $\ell_0^2(n)$. Further, $[q] = \int_0^1 q(x) dx$ denotes the mean value of q .

Proposition 2.1 *For each $n \geq 1$, μ_n and κ_n are real analytic functions on L^2 with L^2 -gradients*

$$\partial \mu_n = g_n^2, \quad \partial \kappa_n = a_n - [a_n] g_n^2,$$

where $a_n = y_1 y_2|_{\mu_n}$. Moreover,

$$\mu_n = n^2 \pi^2 + [q] + \ell^2(n), \quad \kappa_n = \ell_1^2(n),$$

and

$$\begin{aligned} \partial \mu_n &= 1 - \cos 2\pi n x + O\left(\frac{1}{n}\right), & \partial \kappa_n &= \frac{\sin 2\pi n x}{2\pi n} + O\left(\frac{1}{n^2}\right), \\ (\partial \mu_n)' &= 2\pi n \sin 2\pi n x + O(1), & (\partial \kappa_n)' &= \cos 2\pi n x + O\left(\frac{1}{n}\right), \end{aligned}$$

locally uniformly on L^2 .

A similar result holds for the periodic eigenvalues λ_{2n} and λ_{2n-1} , when they are *simple*. Only then they are analytic functions of q and admit unique normalized eigenfunctions f_{2n} and f_{2n-1} . Let $D_n = \{q : \lambda_{2n-1}(q) = \lambda_{2n}(q)\}$.

Proposition 2.2 *For each $n \geq 1$, λ_{2n} and λ_{2n-1} are real analytic functions on $L^2 \setminus D_n$ with L^2 -gradients*

$$\partial \lambda_{2n} = f_{2n}^2, \quad \partial \lambda_{2n-1} = f_{2n-1}^2.$$

Moreover,

$$\lambda_{2n}, \lambda_{2n-1} = n^2 \pi^2 + [q] + \ell^2(n)$$

and

$$\begin{aligned}\partial\lambda_{2n} &= \sin 2\pi n(x - x_n) + O\left(\frac{1}{n}\right), \\ (\partial\lambda_{2n})' &= 2\pi n \cos 2\pi n(x - x_n) + O(1),\end{aligned}$$

with some $0 \leq x_n \leq 2$ locally uniformly on $L^2 \setminus D_n$. The same holds with $2n - 1$ in place of $2n$.

This result can be deduced from the preceding proposition by noting that

$$\lambda_{2n}(q) = \mu_n(q_t)$$

for a properly shifted potential $q_t = q(\cdot + t)$, where t depends on n . Then also $f_{2n}(\cdot, q) = g_n(\cdot - t, q_t)$.

In contrast to the eigenvalues themselves, the quantities

$$\gamma_n^2 = (\lambda_{2n} - \lambda_{2n-1})^2, \quad \tau_n = \frac{1}{2}(\lambda_{2n} + \lambda_{2n-1})$$

are analytic functions of q on all of L^2 . The following is proven in [6].

Proposition 2.3 *For each $n \geq 1$, τ_n and γ_n^2 are real analytic functions on L^2 , such that their L^2 -gradients belong to H^2 . In particular,*

$$\partial\tau_n = 1 + O\left(\frac{1}{n}\right), \quad (\partial\tau_n)' = O(1)$$

locally uniformly on L^2 .

Actually, these three propositions hold on some complex neighbourhood of L^2 independent of n , with D_n as above. See [15] for the μ_n and κ_n , and [6] for the other quantities.

3 Basic Lemma and Proof of Theorem 1.1

We begin with a simple observation about the product of two solutions of the equation $-y'' + qy = \lambda y$ for any q in H^1 , real or complex.

Let $\langle u, v \rangle = \int_0^1 u(x)v(x) dx$, and let $D = d/dx$. Further, let

$$H_0^m = \{u \in H^m : [u] = 0\}.$$

Lemma 3.1 (Basic Lemma) *Let $q \in H^1$, and let f and g be two solutions of $-y'' + qy = \lambda y$, such that either fg is 1-periodic, or g vanishes at 0 and 1. Then*

$$2\lambda \langle fg, h \rangle = \langle fg, Ph \rangle$$

for any $h \in H_0^1$ with $P = -\frac{1}{2}D^2 + 2q + q'I$, where $Ih = \int_0^1 h(x) dx$.

Remark. The right hand side is understood in the weak sense:

$$\langle fg, Ph \rangle = \frac{1}{2} \langle (fg)', h' \rangle + \langle fg, 2qh + q'Ih \rangle.$$

Of course, for $h \in H_0^2$, the identity holds in the strong sense as well.

Proof. One verifies by direct calculation that for any two solutions f and g of $-y'' + qy = \lambda y$ one has

$$L(fg) = 2\lambda D(fg),$$

where $L = -\frac{1}{2}D^3 + qD + Dq$. Hence,

$$2\lambda fg = IL(fg) + c,$$

where $Iu = \int_0^1 u(x) dx$. Pairing both sides of this equation with $h \in H_0^1$, we get

$$2\lambda \langle fg, h \rangle = \langle IL(fg), h \rangle,$$

as the term $\langle c, h \rangle = c[h]$ vanishes.

We have $Ih|_0 = 0$ and $Ih|_1 = [h] = 0$ by the definition of I . Integration by parts thus leads to

$$\begin{aligned} \langle IL(fg), h \rangle &= -\langle L(fg), Ih \rangle \\ &= \frac{1}{2} \left((fg)h' - (fg)'h \right) \Big|_0^1 + \langle fg, LIh \rangle. \end{aligned}$$

If fg is 1-periodic, then the boundary terms clearly vanish, since also h is 1-periodic. If, on the other hand, g vanishes at 0 and 1, then

$$(fg)'h - (fg)h' \Big|_0^1 = fg'h \Big|_0^1 = (fg' - f'g)h \Big|_0^1.$$

The last term vanishes, too, since $fg' - f'g$ is constant by the Wronskian identity. Hence in either case,

$$\langle IL(fg), h \rangle = \langle fg, LIh \rangle.$$

This is the claim, since $LI = P$. ■

As γ_n is differentiable only when it does not vanish, it is convenient to introduce

$$\bar{\partial}\gamma_n = \begin{cases} \partial\gamma_n & \text{when } \gamma_n \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then Theorem 1.1 is contained in the following statement.

Theorem 3.2 *Let $q \in H^m$ with $m \geq 0$. Then (i)*

$$\langle \bar{\partial}\gamma_n, h \rangle = \ell_m^2(n) \|h\|_m$$

for $h \in H^m$, and (ii)

$$\gamma_n = \ell_m^2(n).$$

Both estimates hold locally uniformly in q in a complex neighbourhood of H^m .

Proof. We first show that (ii) follows from (i) for each $m \geq 0$. As the complex neighbourhood of H^m may be described as a union of complex balls centered in H^m , we may connect any q in this neighbourhood with the zero potential by a path

$$q_t = \alpha(t) \operatorname{Re} q + \beta(t) \operatorname{Im} q, \quad 0 \leq t \leq 1,$$

where $\alpha(t) = \min(2t, 1)$ and $\beta(t) = \max(2t - 1, 0)$. By the analyticity of γ_n^2 – see Proposition 2.3 – we then have

$$\begin{aligned} \gamma_n^2(q_s) &= \gamma_n^2(q_t) \Big|_0^s = \int_A \frac{d}{dt} \gamma_n^2(q_t) dt \\ &= 2 \int_A \gamma_n(q_t) \langle \bar{\partial}\gamma_n(q_t), \dot{q}_t \rangle dt \\ &= 2 \int_0^s \gamma_n(q_t) \langle \bar{\partial}\gamma_n(q_t), \dot{q}_t \rangle dt, \end{aligned}$$

where $A = \{t \in [0, s]: \gamma_n(q_t) \neq 0\}$. Hence, by the Schwarz inequality,

$$\begin{aligned} |\gamma_n^2(q_s)|^2 &\leq 4 \int_0^s |\gamma_n^2(q_t)| dt \int_0^s |\langle \bar{\partial}\gamma_n(q_t), \dot{q}_t \rangle|^2 dt \\ &\leq 4 \sup_{0 \leq t \leq 1} |\gamma_n^2(q_t)| \int_0^1 |\langle \bar{\partial}\gamma_n(q_t), \dot{q}_t \rangle|^2 dt. \end{aligned}$$

Taking the supremum over $0 \leq s \leq 1$ on the left hand side and cancelling terms,

$$|\gamma_n^2(q)| \leq \sup_{0 \leq t \leq 1} |\gamma_n^2(q_t)| \leq 4 \int_0^1 |(\bar{\partial}\gamma_n(q_t), \dot{q}_t)|^2 dt.$$

Now (ii) follows from (i), since the estimate of (i) holds uniformly in a neighbourhood of the path q_t by the compactness of the t -interval and $\|\dot{q}_t\|_m \leq 2\|q\|_m$ for $0 \leq t < 1/2$ and $1/2 < t \leq 1$.

Now we prove (i). This is done by induction on m , and we begin with the induction step for $m \geq 2$. It suffices to consider $n \geq 1$ such that $\operatorname{Re} \lambda_{2n-1} > 0$.

Let $h \in H^m$. Since $[\bar{\partial}\gamma_n] = 0$, we have $\langle \bar{\partial}\gamma_n, h \rangle = \langle \bar{\partial}\gamma_n, h_0 \rangle$ for $h_0 = h - [h]$. If $\gamma_n \neq 0$, the Basic Lemma together with Proposition 2.2 then gives

$$\begin{aligned} \langle \bar{\partial}\gamma_n, h \rangle &= \langle f_{2n}^2 - f_{2n-1}^2, h_0 \rangle \\ &= \frac{1}{2\lambda_{2n}} \langle f_{2n}^2, Ph_0 \rangle - \frac{1}{2\lambda_{2n-1}} \langle f_{2n-1}^2, Ph_0 \rangle \\ &= \frac{1}{2\lambda_{2n}} \langle f_{2n}^2 - f_{2n-1}^2, Ph_0 \rangle + \left(\frac{1}{2\lambda_{2n}} - \frac{1}{2\lambda_{2n-1}} \right) \langle f_{2n-1}^2, Ph_0 \rangle. \end{aligned}$$

Hence,

$$\langle \bar{\partial}\gamma_n, h \rangle = \frac{1}{2\lambda_{2n}} \langle \bar{\partial}\gamma_n, Ph_0 \rangle - \frac{\gamma_n}{2\lambda_{2n}\lambda_{2n-1}} \langle f_{2n-1}^2, Ph_0 \rangle. \quad (1)$$

The last identity also holds when $\gamma_n = 0$, where f_{2n-1} could be *any* normalized eigenfunction for $\lambda_{2n} = \lambda_{2n-1}$. So this identity holds everywhere.

We have $Ph_0 \in H^{m-2}$ with $\|Ph_0\|_{m-2} = O(\|h\|_m)$. By the induction hypothesis and $\lambda_{2n} \sim n^2$ we thus obtain

$$\frac{1}{2\lambda_{2n}} \langle \bar{\partial}\gamma_n, Ph_0 \rangle = n^{-2} \ell_{m-2}^2(n) \|Ph_0\|_{m-2} = \ell_m^2(n) \|h\|_m$$

for the first term. As to the second term, note that $\gamma_n = O(n^{-m+2})$ by the induction hypothesis and $f_{2n-1}^2 = O(1)$ to obtain

$$\frac{\gamma_n}{2\lambda_{2n}\lambda_{2n-1}} \langle f_{2n-1}^2, Ph_0 \rangle = O(n^{-m-2}) \|Ph_0\|_{m-2} = \ell_m^2(n) \|h\|_m$$

as well. This completes the induction step.

It remains to establish (i) for $m = 0$ and $m = 1$. For $m = 0$, this is a direct consequence of Proposition 2.2. For $m = 1$, we interpret (1) in the weak sense,

writing

$$\frac{1}{2\lambda_{2n}} \langle \bar{\partial}\gamma_n, Ph_0 \rangle = \frac{1}{2} \langle (\bar{\partial}\gamma_n)', h_0' \rangle + \langle \bar{\partial}\gamma_n, (P - P_0)h_0 \rangle,$$

and similiary

$$\langle f_{2n-1}^2, Ph_0 \rangle = -\frac{1}{2} \langle (f_{2n-1}^2)', h_0' \rangle + \langle f_{2n-1}^2, (P - P_0)h_0 \rangle,$$

where $P_0 = P|_{q=0} = -\frac{1}{2}D^2$. The claim then follows with the asymptotic formulas of Proposition 2.2 for $\partial\gamma_n$ and f_{2n-1}^2 and their derivatives. ■

4 Further Auxiliary Results

In this section we use the approach of the previous section to give new, short proofs of asymptotic estimates for $\tau_n - \mu_n$ and κ_n and their L^2 -gradients introduced in section 2. Let

$$c_n = \cos 2\pi nx,$$

$$s_n = \frac{1}{2\pi n} \sin 2\pi nx,$$

and let $\langle u, v \rangle = \int_0^1 u(x)v(x) dx$ as before.

Proposition 4.1 *Let $q \in H^m$ with $m \geq 0$. Then (i)*

$$\langle \partial\kappa_n, h \rangle = \langle s_n, h \rangle + O\left(\frac{1}{n^{m+2}}\right) \|h\|_m$$

for $h \in H^m$, and (ii)

$$\kappa_n = \langle s_n, q \rangle + O\left(\frac{1}{n^{m+2}}\right).$$

Both estimates hold locally uniformly in q in a complex neighbourhood of H^m .

Proof. Again, (ii) follows from (i). Using the same path q_t as in the proof of the Basic Lemma, we have

$$\begin{aligned} \kappa_n(q) &= \kappa_n(q_t)|_0^1 = \int_0^1 \langle \partial\kappa_n(q_t), \dot{q}_t \rangle dt \\ &= \langle s_n, q \rangle + \int_0^1 \langle \partial\kappa_n(q_t) - s_n, \dot{q}_t \rangle dt, \end{aligned}$$

which gives the result.

To prove (i) by induction, let first $m \geq 2$ and $h \in H^m$. It suffices to consider $n \geq 1$ with $\mu_n > 0$. As, by Proposition 2.1,

$$\partial\kappa_n = a_n - [a_n]g_n^2$$

has mean value zero, $\langle \partial\kappa_n, h \rangle = \langle \partial\kappa_n, h_0 \rangle$ for $h_0 = h - [h]$. The Basic Lemma and the induction hypothesis then give

$$\begin{aligned} \langle \partial\kappa_n, h \rangle &= \frac{1}{2\mu_n} \langle \partial\kappa_n, Ph_0 \rangle \\ &= \frac{1}{2\mu_n} (\langle s_n, Ph_0 \rangle + O(n^{-m}) \|Ph_0\|_{m-2}) \\ &= \frac{1}{2\mu_n} \langle s_n, Ph_0 \rangle + O(n^{-m-2}) \|h\|_m. \end{aligned}$$

Moreover, with $P_0 = P|_{q=0}$ and $\mu_n^o = \mu_n|_{q=0}$,

$$\begin{aligned} \frac{1}{2\mu_n} \langle s_n, Ph_0 \rangle &= \frac{1}{2\mu_n^o} \langle s_n, P_0h_0 \rangle + \frac{\mu_n^o - \mu_n}{2\mu_n^o\mu_n} \langle s_n, P_0h_0 \rangle \\ &\quad + \frac{1}{2\mu_n} \langle s_n, (P - P_0)h_0 \rangle. \end{aligned}$$

The last two terms are again bounded by $O(n^{-m-2}) \|h\|_m$ by standard estimates for Fourier coefficients, while

$$\frac{1}{2\mu_n^o} \langle s_n, P_0h_0 \rangle = -\frac{1}{4\pi^2 n^2} \langle s_n, h'' \rangle = \langle s_n, h \rangle$$

by integration by parts, the boundary terms vanishing by the periodicity of $c_n h$. This completes the induction step.

The claim for $m = 0$ and $m = 1$ follows as in the proof of Theorem 3.2 from the asymptotic formulas for $\partial\kappa_n$ in Proposition 2.1. ■

Proposition 4.2 *Let $q \in H^m$ with $m \geq 0$. Then (i)*

$$\langle \partial(\tau_n - \mu_n), h \rangle = \langle c_n, h \rangle + O\left(\frac{1}{n^{m+1}}\right) \|h\|_m$$

for $h \in H^m$, and (ii)

$$\tau_n - \mu_n = \langle c_n, q \rangle + O\left(\frac{1}{n^{m+1}}\right).$$

Both estimates hold locally uniformly in q in a complex neighbourhood of H^m .

Proof. Again, (ii) follows from (i) as in the previous proof. To prove (i) by induction, we note that also $\partial(\tau_n - \mu_n)$ has mean value zero, whence

$$\langle \partial(\tau_n - \mu_n), h \rangle = \langle \partial(\tau_n - \mu_n), h_0 \rangle$$

for $h_0 = h - [h]$. If $\gamma_n \neq 0$ and n is sufficiently large, the Basic Lemma then gives

$$\begin{aligned} \langle \partial(\tau_n - \mu_n), h \rangle &= \frac{1}{2} \langle \partial\lambda_{2n} + \partial\lambda_{2n-1}, h_0 \rangle - \langle \partial\mu_n, h_0 \rangle \\ &= \frac{1}{2} \sum_{m=2n-1}^{2n} \frac{1}{2\lambda_m} \langle \partial\lambda_m, Ph_0 \rangle - \frac{1}{2\mu_n} \langle \partial\mu_n, Ph_0 \rangle \\ &= \frac{1}{2\tau_n} \langle \partial(\tau_n - \mu_n), Ph_0 \rangle - \frac{\tau_n - \mu_n}{2\tau_n\mu_n} \langle \partial\mu_n, Ph_0 \rangle \\ &\quad + \sum_{m=2n-1}^{2n} \frac{\tau_n - \lambda_m}{4\tau_n\lambda_m} \langle \partial\lambda_m, Ph_0 \rangle. \end{aligned}$$

If $\gamma_n \rightarrow 0$, then the last sum vanishes, since $\tau_n - \lambda_m \rightarrow 0$, while $\partial\lambda_m$ stays bounded in L^2 . Hence the last identity also makes sense for $\gamma_n = 0$, if the sum is understood to be zero.

From this point on, one argues as in the previous proof, using

$$\tau_n - \mu_n = O\left(\frac{1}{n^{m-2}}\right)$$

by the induction hypothesis and standard estimates of Fourier coefficients. The same applies to $\tau_n - \lambda_{2n}$ and $\tau_n - \lambda_{2n-1}$, since a periodic eigenvalue coincides with the corresponding Dirichlet eigenvalue of a properly shifted potential. ■

To summarize the results of this section, let

$$\alpha_n := \tau_n - \mu_n + 2\pi i n \kappa_n$$

and $e_n = e^{2\pi i n x}$.

Theorem 4.3 For each $n \geq 1$, $\operatorname{Re} \alpha_n$ and $\operatorname{Im} \alpha_n$ are real analytic on H^m , with

$$\alpha_n = \langle e_n, q \rangle + O\left(\frac{1}{n^{m+1}}\right).$$

This estimate holds locally uniformly on a complex neighbourhood of H^m .

Remark. With the proofs of Theorems 3.2 and 4.3 we have given an elementary argument for Proposition B.9 in [6, p. 199], stating that

$$\sum_{n \geq 1} n^{2m} (|\gamma_n|^2 + |\tau_n - \mu_n|^2) = O(1)$$

locally uniformly on a small complex neighbourhood of H^m .

5 Proof of Theorem 1.2

It suffices to prove the density of finite gap potentials within the spaces

$$H_0^m = \{q \in H^m : [q] = 0\}$$

of potentials of vanishing mean value, since adding a constant to a potential just shifts the entire spectrum, leaving the gap lengths unchanged.

Rather than the gap lengths, however, we consider the quantities α_n introduced above in view of the following simple observation.

Lemma 5.1 For q in L_0^2 and any $n \geq 1$,

$$\gamma_n(q) = 0 \quad \text{iff} \quad \alpha_n(q) = 0.$$

Proof. Fix q and n . If $\gamma_n = 0$, then $\mu_n = \tau_n$, and the n -th Dirichlet eigenfunction g_n is also a periodic or anti-periodic eigenfunction. But then

$$|y_2'(1, \mu_n)| = 1,$$

whence also $\kappa_n = 0$, and thus $\alpha_n = 0$.

Conversely, if $\alpha_n = 0$, then $\kappa_n = 0$ implies that g_n is a periodic or anti-periodic eigenfunction, hence μ_n is also a periodic eigenvalue. Since in addition $\mu_n = \tau_n$, the corresponding gap must be collapsed, whence $\gamma_n = 0$. ■

Consider now the map

$$A: H_0^m \rightarrow \mathfrak{h}^m, \quad q \mapsto (\alpha_n(q))_{n \geq 1},$$

where \mathfrak{h}^m is the Hilbert space of all complex sequences $v = (v_n)_{n \geq 1}$ with

$$\|v\|_m^2 = \sum_{n \geq 1} n^{2m} |v_n|^2 < \infty.$$

By Theorem 4.3 and Theorem A.5 in [6] this map is analytic. By the previous lemma, q is a finite gap potential, iff all but finitely many coordinates of $A(q)$ vanish.

To prove Theorem 1.2, however, it is rather more convenient to consider the map

$$G = A \circ \Phi: \mathfrak{h}^m \rightarrow \mathfrak{h}^m,$$

where

$$\Phi: \mathfrak{h}^m \rightarrow H_0^m, \quad (\xi_n)_{n \geq 1} \mapsto 2 \operatorname{Re} \sum_{n \geq 1} \xi_n e^{2\pi i n x}$$

is the inverse of the restriction of the discrete Fourier transform to H_0^m . Since Φ is a linear isomorphism it suffices to prove the following statement, which also contains the statement made in Remark 2 in [6, p. 206].

Proposition 5.2 *For ξ in a dense subset of \mathfrak{h}^m , with $m \geq 0$, all but finitely many coordinates of $G(\xi)$ vanish.*

Proof. In view of Theorem 4.3, the map G is *real analytic*, when considered as a map

$$(\operatorname{Re} \xi, \operatorname{Im} \xi) \mapsto (\operatorname{Re} G(\xi), \operatorname{Im} G(\xi)).$$

It is of the form $I + K$, where K maps \mathfrak{h}^m into a smaller space $\mathfrak{h}^{m+\sigma}$, $0 < \sigma < 1/2$. It follows with Cauchy's inequality that on some ball around any given point in \mathfrak{h}^m , the Jacobian dK is uniformly bounded as a linear map $\mathfrak{h}^m \rightarrow \mathfrak{h}^{m+\sigma}$. Consequently,

$$\|T_N dK\|_m \leq \frac{1}{2}$$

on the same ball for all sufficiently large N in the operator norm on \mathfrak{h}^m , where T_N denotes the projection onto all *except* the first N coordinates in \mathfrak{h}^m .

Now fix ξ^0 in \mathfrak{h}^m , and let $\varepsilon > 0$ be so small that the preceding estimate holds on the 4ε -ball B around ξ^0 for all sufficiently large N . We may then fix N so large

that also

$$\|T_N G(\xi^0)\|_m < \varepsilon.$$

Writing $\xi = \xi_N + \zeta_N$ with $\zeta_N = T_N \xi$ we then have

$$T_N G(\xi) = T_N G(\xi_N + \zeta_N) = \zeta_N + T_N K(\xi_N + \zeta_N)$$

with

$$\|d_{\zeta_N} T_N K\|_m \leq \frac{1}{2}$$

uniformly on B . The map

$$\zeta_N \mapsto \zeta_N + T_N K(\xi_N^0 + \zeta_N)$$

is thus a local diffeomorphism, and by the inverse function theorem the image of the ball $\|\zeta_N\|_m < 4\varepsilon$ under this map covers a ball of radius 2ε around $T_N G(\xi^0)$. Consequently, in view of $\|T_N G(\xi^0)\|_m < \varepsilon$, there exists $\xi^s = \xi_N^0 + \zeta_N^s$ with

$$\|\xi^s - \xi^0\|_m = \|\zeta_N^s - \zeta_N^0\|_m < 4\varepsilon$$

such that $T_N G(\xi^s) = 0$. Since $\varepsilon > 0$ can be chosen arbitrarily small, this proves the claim. ■

Remark. The proof incidentally shows that there exists a finite gap potential with any finite number of Fourier coefficients prescribed arbitrarily.

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