IMRN International Mathematics Research Notices
2004, No. 50

A Note on Gaps of Hill's Equation

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1 Results

We consider the differential operator

$$L = -\frac{d^2}{dx^2} + q, \quad q \in L^2 = L^2\left(S^1, \mathbb{R}\right) \tag{1.1}$$

on the interval $[0, 1]$ endowed with periodic or antiperiodic boundary conditions

$$y(0) = y(1), \qquad y'(0) = y'(1) \tag{1.2}$$

or

$$y(0) = -y(1), \qquad y'(0) = -y'(1). \tag{1.3}$$

The corresponding differential equation

$$-y'' + qy = \lambda y \tag{1.4}$$

is also known as *Hill's equation with potential* q.

It is well known that the spectrum of L is pure point and consists of an unbounded sequence of *periodic eigenvalues*

$$\lambda_0(q) < \lambda_1(q) \le \lambda_2(q) < \lambda_3(q) \le \lambda_4(q) < \cdots . \tag{1.5}$$

Received 15 September 2003. Revision received 8 April 2004.
Communicated by Kurt Johansson.

Equality or inequality may occur in every place with a "$\le$"-sign, and one speaks of the *gaps* $(\lambda_{2n-1}(q), \lambda_{2n}(q))$ of the potential q and its *gap lengths*

$$\gamma_n(q) = \lambda_{2n}(q) - \lambda_{2n-1}(q), \quad n \ge 1. \tag{1.6}$$

If some gap length is zero, one speaks of a *collapsed gap*, otherwise of an *open gap*.

The purpose of this paper is to give new, short proofs of two facts relating these gap lengths to the regularity of the potential q. To formulate these results, denote by $H^m = H^m(S^1)$ the Sobolev space of m-times weakly differentiable functions of period 1. That is,

$$H^m = \{u \in L^2(S^1, \mathbb{R}) : \|u\|_m < \infty\}, \tag{1.7}$$

where $\|u\|_m^2 = |\hat{u}(0)|^2 + \sum_{n\neq 0} n^{2m}|\hat{u}(n)|^2$ is defined in terms of the discrete Fourier transform $\hat{u}$ of u.

The following result was first proven by Marčenko and Ostrowskiĭ [12], using inverse spectral theory. Their approach was later simplified by Garnett and Trubowitz [3] and generalized in [7]. For a more elementary proof, see also [4, 5].

Theorem 1.1. The gap lengths satisfy

$$\sum_{n\ge 1} n^{2m}\gamma_n^2 < \infty \tag{1.8}$$

locally uniformly on H^m for any $m \ge 0$. □

In fact, Marčenko and Ostrowskiĭ [12] also prove the converse statement: if the gaps of a given potential in H^0 are as above, then this potential is in H^m. For further results in this direction, see also [3, 7, 8].

The second result concerns the density of finite-gap potentials, which are potentials with only a finite number of open gaps.

Theorem 1.2. Finite-gap potentials are dense in H^m for any $m \ge 0$. □

This result was conjectured by Novikov [14] (see also Lax [9]) and first proven by Marčenko and Ostrovskiĭ [12]. See also [3, 7, 10, 11] and others, for example, [13]. While these approaches use inverse spectral theory, our proof uses only asymptotic properties of some spectral data. In this respect, the first proof sans inverse spectral theory appeared in [1] for the case $m = 0$.

We point out that Theorems 1.1 and 1.2 are used in the proof of the normal form theorem for KdV in [6], asserting that KdV admits Birkhoff coordinates in any Sobolev space H^m, $m \geq 0$. The case $m = 0$ of these two theorems is treated in detail in [6], while the case $m \geq 1$ is quoted from other sources. In this paper we supply a proof for the case $m \geq 1$ along the same lines as for $m = 0$—as it should have appeared in [6].

The rest of this paper is devoted to new proofs of these two results. Indeed, we also show that Theorem 1.1 holds in some complex neighbourhood of H^m for each $m \geq 0$, and that there is a similar result for quantities involving Dirichlet eigenvalues.

2 Some background

Denote by y_1, y_2 the fundamental solution of $-y'' + qy = \lambda y$ satisfying

$$\begin{aligned} y_1(0,\lambda,q) &= 1, & y_2(0,\lambda,q) &= 0, \\ y_1'(0,\lambda,q) &= 0, & y_2'(0,\lambda,q) &= 1. \end{aligned} \tag{2.1}$$

The spectrum of the operator

$$L = -\frac{d^2}{dx^2} + q, \quad q \in L^2, \tag{2.2}$$

endowed with Dirichlet boundary conditions, is called the *Dirichlet spectrum of* q and coincides with the zero set of the entire function $y_2(1,\cdot,q)$. It is an unbounded sequence of *Dirichlet eigenvalues*

$$\mu_1(q) < \mu_2(q) < \mu_3(q) < \cdots, \tag{2.3}$$

which are all simple. With each eigenvalue one can associate a unique *Dirichlet eigenfunction*

$$g_n = \left.\frac{y_2}{\|y_2\|}\right|_{\lambda=\mu_n}, \quad n \geq 1. \tag{2.4}$$

Besides the μ_n, we also need to consider the quantities

$$\kappa_n(q) = \log(-1)^n y_2'\big(1, \mu_n(q), q\big), \quad n \geq 1, \tag{2.5}$$

which measure the terminal velocities of the eigenfunctions.

The following facts are proven in [15]. We write $\ell^2_m(n)$ for the nth term of a generic sequence $x = (x_n)_{n\geq 1}$ with

$$\sum_{n\geq 1} n^{2m}|x_n|^2 < \infty, \tag{2.6}$$

and $\ell^2(n)$ for $\ell^2_0(n)$. Further, $[q] = \int_0^1 q(x)dx$ denotes the mean value of q.

Proposition 2.1. For each $n \geq 1$, μ_n and κ_n are real analytic functions on L^2 with L^2-gradients

$$\partial\mu_n = g_n^2, \qquad \partial\kappa_n = a_n - [a_n]g_n^2, \tag{2.7}$$

where $a_n = y_1 y_2|_{\mu_n}$. Moreover,

$$\mu_n = n^2\pi^2 + [q] + \ell^2(n), \qquad \kappa_n = \ell^2_1(n), \tag{2.8}$$

and

$$\begin{aligned} \partial\mu_n &= 1 - \cos 2\pi nx + O\left(\frac{1}{n}\right), &\qquad \partial\kappa_n &= \frac{\sin 2\pi nx}{2\pi n} + O\left(\frac{1}{n^2}\right), \\ (\partial\mu_n)' &= 2\pi n \sin 2\pi nx + O(1), &\qquad (\partial\kappa_n)' &= \cos 2\pi nx + O\left(\frac{1}{n}\right) \end{aligned} \tag{2.9}$$

locally uniformly on L^2. □

A similar result holds for the periodic eigenvalues λ_{2n} and λ_{2n-1}, when they are *simple*. Only then are they analytic functions of q and do they admit unique normalized eigenfunctions f_{2n} and f_{2n-1}. Let $D_n = \{q : \lambda_{2n-1}(q) = \lambda_{2n}(q)\}$.

Proposition 2.2. For each $n \geq 1$, λ_{2n} and λ_{2n-1} are real analytic functions on $L^2 \setminus D_n$ with L^2-gradients

$$\partial\lambda_{2n} = f_{2n}^2, \qquad \partial\lambda_{2n-1} = f_{2n-1}^2. \tag{2.10}$$

Moreover,

$$\lambda_{2n}, \lambda_{2n-1} = n^2\pi^2 + [q] + \ell^2(n) \tag{2.11}$$

and

$$\begin{aligned}\partial\lambda_{2n} &= \sin 2\pi n(x - x_n) + O\left(\frac{1}{n}\right),\\ (\partial\lambda_{2n})' &= 2\pi n \cos 2\pi n(x - x_n) + O(1),\end{aligned} \tag{2.12}$$

with some $0 \le x_n \le 2$ locally uniformly on $L^2 \setminus D_n$. The same holds with $2n-1$ in place of $2n$. $\square$

This result can be deduced from the preceding proposition by noting that

$$\lambda_{2n}(q) = \mu_n(q_t) \tag{2.13}$$

for a properly shifted potential $q_t = q(\cdot + t)$, where t depends on n. Then also $f_{2n}(\cdot, q) = g_n(\cdot, q_t)$.

In contrast to the eigenvalues themselves, the quantities

$$\gamma_n^2 = (\lambda_{2n} - \lambda_{2n-1})^2, \qquad \tau_n = \frac{1}{2}(\lambda_{2n} + \lambda_{2n-1}) \tag{2.14}$$

are analytic functions of q on all of L^2. The following is proven in [6].

Proposition 2.3. For each $n \ge 1$, τ_n and γ_n^2 are real analytic functions on L^2, such that their L^2-gradients belong to H^2. In particular,

$$\partial\tau_n = 1 + O\left(\frac{1}{n}\right), \qquad (\partial\tau_n)' = O(1) \tag{2.15}$$

locally uniformly on L^2. $\square$

Actually, these three propositions hold on some complex neighbourhood of L^2 independent of n, with D_n as above. See [15] for the μ_n and κ_n, and [6] for the other quantities.

3 Basic lemma and proof of Theorem 1.1

We begin with a simple observation about the product of two solutions of the equation $-y'' + qy = \lambda y$ for any q in H^1, real or complex.

Let $\langle u, v\rangle = \int_0^1 u(x)v(x)\,dx$, and let $D = (d/dx)$. Further, let

$$H_0^m = \{u \in H^m : [u] = 0\}. \tag{3.1}$$

Lemma 3.1 (basic lemma)**.** Let f and g be two solutions of $-y'' + qy = \lambda y$ with $q \in H^1$, such that either fg is 1-periodic or g vanishes at 0 and 1. Then

$$2\lambda\langle fg, h\rangle = \langle fg, Ph\rangle \tag{3.2}$$

for any $h \in H_0^1$ with $P = -(1/2)D^2 + 2q + q'I$, where $Ih = \int_0 h(x)dx$. □

Remark 3.2. The right-hand side is understood in the weak sense:

$$\langle fg, Ph\rangle = \frac{1}{2}\big\langle (fg)', h'\big\rangle + \big\langle fg, 2qh + q'Ih\big\rangle. \tag{3.3}$$

Of course, for $h \in H_0^2$, the identity holds in the strong sense as well.

Proof. One verifies by direct calculation that for any two solutions f and g of the equation $-y'' + qy = \lambda y$, one has

$$L(fg) = 2\lambda D(fg), \tag{3.4}$$

where $L = (-1/2)D^3 + qD + Dq$. Hence,

$$2\lambda fg = IL(fg) + c, \tag{3.5}$$

where $Iu = \int_0 u(x)dx$. Pairing both sides of this equation with $h \in H_0^1$, we get

$$2\lambda\langle fg, h\rangle = \big\langle IL(fg), h\big\rangle, \tag{3.6}$$

as the term $\langle c, h\rangle = c[h]$ vanishes.

We have $Ih|_0 = 0$ and $Ih|_1 = [h] = 0$ by the definition of I. Integration by parts thus leads to

$$\big\langle IL(fg), h\big\rangle = -\big\langle L(fg), Ih\big\rangle = \frac{1}{2}\big((fg)h' - (fg)'h\big)\Big|_0^1 + \langle fg, LIh\rangle. \tag{3.7}$$

If fg is 1-periodic, then the boundary terms clearly vanish, since also h is 1-periodic. If, on the other hand, g vanishes at 0 and 1, then

$$(fg)'h - (fg)h'\Big|_0^1 = fg'h\Big|_0^1 = \big(fg' - f'g\big)h\Big|_0^1. \tag{3.8}$$

The last term vanishes too since $fg' - f'g$ is constant by the Wronskian identity. Hence in either case,

$$\big\langle IL(fg), h\big\rangle = \langle fg, LIh\rangle. \tag{3.9}$$

This is the claim, since $LI = P$. ■

As γ_n is differentiable only when it does not vanish, it is convenient to introduce

$$\eth\gamma_n = \begin{cases} \partial\gamma_n & \text{when } \gamma_n \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Then Theorem 1.1 is contained in the following statement.

Theorem 3.3. Let $q \in H^m$ with $m \geq 0$. Then

$$\langle \eth\gamma_n, h\rangle = \ell_m^2(n)\|h\|_m \tag{3.11}$$

for $h \in H^m$, and

$$\gamma_n = \ell_m^2(n). \tag{3.12}$$

Both estimates hold locally uniformly in q in a complex neighbourhood of H^m. □

Proof. We first show that (3.12) follows from (3.11) for each $m \geq 0$. As the complex neighbourhood of H^m may be described as a union of complex balls centered in H^m, we may connect any q in this neighbourhood with the zero potential by a path

$$q_t = \alpha(t)\,\mathrm{Re}\,q + \beta(t)\,\mathrm{Im}\,q, \quad 0 \leq t \leq 1, \tag{3.13}$$

where $\alpha(t) = \min(2t, 1)$ and $\beta(t) = \max(2t-1, 0)$. By the analyticity of γ_n^2—see Proposition 2.3—we then have

$$\begin{aligned} \gamma_n^2(q_s) = \gamma_n^2(q_t)\big|_0^s &= \int_A \frac{d}{dt}\gamma_n^2(q_t)\,dt \\ &= 2\int_A \gamma_n(q_t)\langle \eth\gamma_n(q_t), \dot{q}_t\rangle dt \\ &= 2\int_0^s \gamma_n(q_t)\langle \eth\gamma_n(q_t), \dot{q}_t\rangle dt, \end{aligned} \tag{3.14}$$

where $A = \{t \in [0, s] : \gamma_n(q_t) \neq 0\}$. Hence, by the Schwarz inequality,

$$\begin{aligned} \left|\gamma_n^2(q_s)\right|^2 &\leq 4\int_0^s \left|\gamma_n^2(q_t)\right| dt \int_0^s \left|\langle \eth\gamma_n(q_t), \dot{q}_t\rangle\right|^2 dt \\ &\leq 4 \sup_{0\leq t\leq 1} \left|\gamma_n^2(q_t)\right| \int_0^1 \left|\langle \eth\gamma_n(q_t), \dot{q}_t\rangle\right|^2 dt. \end{aligned} \tag{3.15}$$

Taking the supremum over $0 \le s \le 1$ on the left-hand side and cancelling terms,

$$\left|\gamma_n^2(q)\right| \le \sup_{0\le t\le 1} \left|\gamma_n^2(q_t)\right| \le 4\int_0^1 \left|\left\langle \eth\gamma_n(q_t), \dot q_t\right\rangle\right|^2 dt. \tag{3.16}$$

Now (3.12) follows from (3.11) since the estimate of (3.11) holds uniformly in a neighbourhood of the path q_t by the compactness of the t-interval and $\|\dot q_t\|_m \le 2\|q\|_m$ for $0 \le t < 1/2$ and $1/2 < t \le 1$.

Now we prove (3.11). This is done by induction on m, and we begin with the induction step for $m \ge 2$. It suffices to consider $n \ge 1$ such that $\operatorname{Re}\lambda_{2n-1} > 0$.

Let $h \in H^m$. Since $[\eth\gamma_n] = 0$, we have $\langle \eth\gamma_n, h\rangle = \langle \eth\gamma_n, h_0\rangle$ for $h_0 = h - [h]$. If $\gamma_n \ne 0$, Lemma 3.1, together with Proposition 2.2, then gives

$$\begin{aligned}\left\langle \eth\gamma_n, h\right\rangle &= \left\langle f_{2n}^2 - f_{2n-1}^2, h_0\right\rangle \\ &= \frac{1}{2\lambda_{2n}}\left\langle f_{2n}^2, Ph_0\right\rangle - \frac{1}{2\lambda_{2n-1}}\left\langle f_{2n-1}^2, Ph_0\right\rangle \\ &= \frac{1}{2\lambda_{2n}}\left\langle f_{2n}^2 - f_{2n-1}^2, Ph_0\right\rangle + \left(\frac{1}{2\lambda_{2n}} - \frac{1}{2\lambda_{2n-1}}\right)\left\langle f_{2n-1}^2, Ph_0\right\rangle.\end{aligned} \tag{3.17}$$

Hence,

$$\left\langle \eth\gamma_n, h\right\rangle = \frac{1}{2\lambda_{2n}}\left\langle \eth\gamma_n, Ph_0\right\rangle - \frac{\gamma_n}{2\lambda_{2n}\lambda_{2n-1}}\left\langle f_{2n-1}^2, Ph_0\right\rangle. \tag{3.18}$$

The last identity also holds when $\gamma_n = 0$, where f_{2n-1} could be *any* normalized eigenfunction for $\lambda_{2n} = \lambda_{2n-1}$. So this identity holds everywhere.

We have $Ph_0 \in H^{m-2}$ with $\|Ph_0\|_{m-2} = O(\|h\|_m)$. By the induction hypothesis and $\lambda_{2n} \sim n^2$, we thus obtain

$$\frac{1}{2\lambda_{2n}}\left\langle \eth\gamma_n, Ph_0\right\rangle = n^{-2}\ell_{m-2}^2(n)\left\|Ph_0\right\|_{m-2} = \ell_m^2(n)\|h\|_m \tag{3.19}$$

for the first term. As to the second term, note that $\gamma_n = O(n^{-m+2})$ by the induction hypothesis and $f_{2n-1}^2 = O(1)$ to obtain

$$\frac{\gamma_n}{2\lambda_{2n}\lambda_{2n-1}}\left\langle f_{2n-1}^2, Ph_0\right\rangle = O\left(n^{-m-2}\right)\left\|Ph_0\right\|_{m-2} = \ell_m^2(n)\|h\|_m \tag{3.20}$$

as well. This completes the induction step.

It remains to establish (3.11) for $m = 0$ and $m = 1$. For $m = 0$, this is a direct consequence of Proposition 2.2. For $m = 1$, we interpret (3.18) in the weak sense, writing

$$\frac{1}{2\lambda_{2n}}\left\langle \eth\gamma_n, Ph_0\right\rangle = \frac{1}{2}\left\langle \left(\eth\gamma_n\right)', h_0'\right\rangle + \left\langle \eth\gamma_n, \left(P - P_0\right)h_0\right\rangle, \tag{3.21}$$

and similarly,

$$\left\langle f_{2n-1}^2, Ph_0\right\rangle = -\frac{1}{2}\left\langle \left(f_{2n-1}^2\right)', h_0'\right\rangle + \left\langle f_{2n-1}^2, (P - P_0)h_0\right\rangle, \tag{3.22}$$

where $P_0 = P|_{q=0} = -(1/2)D^2$. The claim then follows with the asymptotic formulas of Proposition 2.2 for $\partial\gamma_n$ and f_{2n-1}^2 and their derivatives. ■

4 Further auxiliary results

In this section we use the approach of the previous section to give new, short proofs of asymptotic estimates for $\tau_n - \mu_n$ and κ_n and their L^2-gradients introduced in Section 2. Let

$$\begin{aligned} c_n &= \cos 2\pi n x, \\ s_n &= \frac{1}{2\pi n}\sin 2\pi n x, \end{aligned} \tag{4.1}$$

and let $\langle u, v\rangle = \int_0^1 u(x)v(x)dx$ as before.

Proposition 4.1. Let $q \in H^m$ with $m \geq 0$. Then

$$\left\langle \partial\kappa_n, h\right\rangle = \left\langle s_n, h\right\rangle + O\left(\frac{1}{n^{m+2}}\right)\|h\|_m \tag{4.2}$$

for $h \in H^m$, and

$$\kappa_n = \left\langle s_n, q\right\rangle + O\left(\frac{1}{n^{m+2}}\right). \tag{4.3}$$

Both estimates hold locally uniformly in q in a complex neighbourhood of H^m. □

Proof. Again, (4.3) follows from (4.2). Using the same path q_t as in the proof of Lemma 3.1, we have

$$\begin{aligned} \kappa_n(q) = \kappa_n(q_t)\big|_0^1 &= \int_0^1 \langle \partial\kappa_n(q_t), \dot{q}_t\rangle dt \\ &= \left\langle s_n, q\right\rangle + \int_0^1 \langle \partial\kappa_n(q_t) - s_n, \dot{q}_t\rangle dt, \end{aligned} \tag{4.4}$$

which gives the result.

To prove (4.2) by induction, first let $m \geq 2$ and $h \in H^m$. It suffices to consider $n \geq 1$ with $\mu_n > 0$. As, by Proposition 2.1,

$$\partial\kappa_n = a_n - [a_n]g_n^2 \tag{4.5}$$

has mean value zero, $\langle \partial\kappa_n, h\rangle = \langle \partial\kappa_n, h_0\rangle$ for $h_0 = h - [h]$. Lemma 3.1 and the induction hypothesis then give

$$\begin{aligned}\langle \partial\kappa_n, h\rangle &= \frac{1}{2\mu_n}\langle \partial\kappa_n, Ph_0\rangle \\ &= \frac{1}{2\mu_n}\Big(\langle s_n, Ph_0\rangle + O(n^{-m})\|Ph_0\|_{m-2}\Big) \\ &= \frac{1}{2\mu_n}\langle s_n, Ph_0\rangle + O(n^{-m-2})\|h\|_m.\end{aligned} \tag{4.6}$$

Moreover, with $P_0 = P|_{q=0}$ and $\mu_n^o = \mu_n|_{q=0}$,

$$\begin{aligned}\frac{1}{2\mu_n}\langle s_n, Ph_0\rangle = \frac{1}{2\mu_n^o}\langle s_n, P_0h_0\rangle + \frac{\mu_n^o - \mu_n}{2\mu_n^o\mu_n}\langle s_n, P_0h_0\rangle \\ + \frac{1}{2\mu_n}\langle s_n, (P - P_0)h_0\rangle.\end{aligned} \tag{4.7}$$

The last two terms are again bounded by $O(n^{-m-2})\|h\|_m$ by standard estimates for Fourier coefficients, while

$$\frac{1}{2\mu_n^o}\langle s_n, P_0h_0\rangle = -\frac{1}{4\pi^2n^2}\langle s_n, h''\rangle = \langle s_n, h\rangle \tag{4.8}$$

by integration by parts, the boundary terms vanishing by the periodicity of c_nh. This completes the induction step.

The claim for $m = 0$ and $m = 1$ follows as in the proof of Theorem 3.3 from the asymptotic formulas for $\partial\kappa_n$ in Proposition 2.1. ■

Proposition 4.2. Let $q \in H^m$ with $m \geq 0$. Then

$$\langle \partial(\tau_n - \mu_n), h\rangle = \langle c_n, h\rangle + O\left(\frac{1}{n^{m+1}}\right)\|h\|_m \tag{4.9}$$

for $h \in H^m$, and

$$\tau_n - \mu_n = \langle c_n, q\rangle + O\left(\frac{1}{n^{m+1}}\right). \tag{4.10}$$

Both estimates hold locally uniformly in q in a complex neighbourhood of H^m. □

Proof. Again, (4.10) follows from (4.9) as in the previous proof. To prove (4.9) by induction, we note that also $\partial(\tau_n - \mu_n)$ has mean value zero, whence

$$\langle \partial(\tau_n - \mu_n), h\rangle = \langle \partial(\tau_n - \mu_n), h_0\rangle \tag{4.11}$$

for $h_0 = h - [h]$. If $\gamma_n \neq 0$ and n is sufficiently large, Lemma 3.1 then gives

$$\begin{aligned}\langle \partial(\tau_n - \mu_n), h\rangle &= \frac{1}{2}\langle \partial\lambda_{2n} + \partial\lambda_{2n-1}, h_0\rangle - \langle \partial\mu_n, h_0\rangle \\ &= \frac{1}{2}\sum_{m=2n-1}^{2n} \frac{1}{2\lambda_m}\langle \partial\lambda_m, Ph_0\rangle - \frac{1}{2\mu_n}\langle \partial\mu_n, Ph_0\rangle \\ &= \frac{1}{2\tau_n}\langle \partial(\tau_n - \mu_n), Ph_0\rangle - \frac{\tau_n - \mu_n}{2\tau_n\mu_n}\langle \partial\mu_n, Ph_0\rangle \\ &\quad + \sum_{m=2n-1}^{2n} \frac{\tau_n - \lambda_m}{4\tau_n\lambda_m}\langle \partial\lambda_m, Ph_0\rangle.\end{aligned} \tag{4.12}$$

If $\gamma_n \to 0$, then the last sum vanishes, since $\tau_n - \lambda_m \to 0$, while $\partial\lambda_m$ stays bounded in L^2. Hence the last identity also makes sense for $\gamma_n = 0$, if the sum is understood to be zero.

From this point on, one argues as in the previous proof, using

$$\tau_n - \mu_n = O\left(\frac{1}{n^{m-2}}\right) \tag{4.13}$$

by the induction hypothesis and standard estimates of Fourier coefficients. The same applies to $\tau_n - \lambda_{2n}$ and $\tau_n - \lambda_{2n-1}$, since a periodic eigenvalue coincides with the corresponding Dirichlet eigenvalue of a properly shifted potential. ■

To summarize the results of this section, let

$$\alpha_n := \tau_n - \mu_n + 2\pi i n \kappa_n \tag{4.14}$$

and $e_n = e^{2\pi i n x}$.

Theorem 4.3. For each $n \geq 1$, $\operatorname{Re}\alpha_n$ and $\operatorname{Im}\alpha_n$ are real analytic on H^m, with

$$\alpha_n = \langle e_n, q\rangle + O\left(\frac{1}{n^{m+1}}\right). \tag{4.15}$$

This estimate holds locally uniformly on a complex neighbourhood of H^m. □

Remark 4.4. With the proofs of Theorems 3.3 and 4.3 we have given an elementary argument for [6, Proposition B.9, page 199], stating that

$$\sum_{n\geq 1} n^{2m}\left(|\gamma_n|^2 + |\tau_n - \mu_n|^2\right) = O(1) \tag{4.16}$$

locally uniformly on a small complex neighbourhood of H^m.

5 Proof of Theorem 1.2

It suffices to prove the density of finite-gap potentials within the spaces

$$H_0^m = \{q \in H^m : [q] = 0\} \tag{5.1}$$

of potentials of vanishing mean value, since adding a constant to a potential just shifts the entire spectrum, leaving the gap lengths unchanged.

Rather than the gap lengths, however, we consider the quantities α_n introduced above in view of the following simple observation.

Lemma 5.1. For q in L_0^2 and any $n \geq 1$,

$$\gamma_n(q) = 0 \Longleftrightarrow \alpha_n(q) = 0. \tag{5.2}$$

□

Proof. Fix q and n. If $\gamma_n = 0$, then $\mu_n = \tau_n$, and the nth Dirichlet eigenfunction g_n is also a periodic or antiperiodic eigenfunction. But then

$$|y_2'(1, \mu_n)| = 1, \tag{5.3}$$

whence also $\kappa_n = 0$, and thus $\alpha_n = 0$.

Conversely, if $\alpha_n = 0$, then $\kappa_n = 0$ implies that g_n is a periodic or antiperiodic eigenfunction, hence μ_n is also a periodic eigenvalue. Since in addition $\mu_n = \tau_n$, the corresponding gap must be collapsed, whence $\gamma_n = 0$. ■

Consider now the map

$$A : H_0^m \longrightarrow \mathscr{h}^m, \quad q \longmapsto (\alpha_n(q))_{n \geq 1}, \tag{5.4}$$

where $\mathscr{h}^m$ is the Hilbert space of all *complex* sequences $v = (v_n)_{n \geq 1}$ with

$$\|v\|_m^2 = \sum_{n \geq 1} n^{2m} |v_n|^2 < \infty. \tag{5.5}$$

By Theorem 4.3 and [6, Theorem A.5] this map is analytic. By the previous lemma, q is a finite-gap potential, if and only if all but finitely many coordinates of $A(q)$ vanish.

To prove Theorem 1.2, however, it is rather more convenient to consider the map

$$G = A \circ \Phi : \mathscr{h}^m \longrightarrow \mathscr{h}^m, \tag{5.6}$$

where

$$\Phi : \mathfrak{h}^m \longrightarrow H_0^m, \quad (\xi_n)_{n\geq 1} \longmapsto 2\,\mathrm{Re} \sum_{n\geq 1} \xi_n e^{2\pi i n x}, \tag{5.7}$$

is the inverse of the restriction of the discrete Fourier transform to H_0^m. Since Φ is a linear isomorphism, it suffices to prove the following statement, which also contains the statement made in [6, Remark 2, page 206].

Proposition 5.2. For ξ in a dense subset of $\mathfrak{h}^m$, with $m \geq 0$, all but finitely many coordinates of $G(\xi)$ vanish. □

Proof. In view of Theorem 4.3, the map G is *real analytic* when considered as a map

$$(\mathrm{Re}\,\xi, \mathrm{Im}\,\xi) \longmapsto \big(\mathrm{Re}\,G(\xi), \mathrm{Im}\,G(\xi)\big). \tag{5.8}$$

It is of the form $I + K$, where K maps $\mathfrak{h}^m$ into a smaller space $\mathfrak{h}^{m+\sigma}$, $0 < \sigma < 1/2$. It follows with Cauchy's inequality that on some ball around any given point in $\mathfrak{h}^m$, the Jacobian dK is uniformly bounded as a linear map $\mathfrak{h}^m \to \mathfrak{h}^{m+\sigma}$. Consequently,

$$\left\| T_N \, dK \right\|_m \leq \frac{1}{2} \tag{5.9}$$

on the same ball for all sufficiently large N in the operator norm on $\mathfrak{h}^m$, where T_N denotes the projection onto all *except* the first N coordinates in $\mathfrak{h}^m$.

Now fix ξ^o in $\mathfrak{h}^m$ and let $\varepsilon > 0$ be so small that the preceding estimate holds on the 4ε-ball B around ξ^o for all sufficiently large N. We may then fix N so large that also

$$\left\| T_N \, G\big(\xi^o\big) \right\|_m < \varepsilon. \tag{5.10}$$

Writing $\xi = \xi_N + \zeta_N$ with $\zeta_N = T_N \xi$, we then have

$$T_N \, G(\xi) = T_N \, G\big(\xi_N + \zeta_N\big) = \zeta_N + T_N \, K\big(\xi_N + \zeta_N\big) \tag{5.11}$$

with

$$\left\| d_{\zeta_N} T_N \, K \right\|_m \leq \frac{1}{2} \tag{5.12}$$

uniformly on B. The map

$$\zeta_N \longmapsto \zeta_N + T_N \, K\big(\xi_N^o + \zeta_N\big) \tag{5.13}$$

is thus a local diffeomorphism, and by the inverse function theorem, the image of the ball $\|\zeta_N\|_m < 4\varepsilon$ under this map covers a ball of radius 2ε around $T_N G(\xi^o)$. Consequently, in view of $\|T_N G(\xi^o)\|_m < \varepsilon$, there exists $\xi^s = \xi_N^o + \zeta_N^s$ with

$$\left\|\xi^s - \xi^o\right\|_m = \left\|\zeta_N^s - \zeta_N^o\right\|_m < 4\varepsilon \tag{5.14}$$

such that $T_N G(\xi^s) = 0$. Since $\varepsilon > 0$ can be chosen arbitrarily small, this proves the claim. ∎

Remark 5.3. The proof incidentally shows that there exists a finite-gap potential with any finite number of Fourier coefficients prescribed arbitrarily.

Acknowledgment

The second author was supported in part by the Swiss National Science Foundation and the European Research Training Network HPRN-CT-1999-00118.

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