A General Infinite Dimensional KAM-Theorem

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ABSTRACT. A general perturbation theory of Kolmogorov-Arnold-Moser type is described concerning the existence of infinite dimensional invariant tori in Hamiltonian systems.

1. THE SET UP

Let Λ be an arbitrary lattice or a subset thereof. Each site λ of this lattice is occupied by a single harmonic oscillator whose configuration is described by a single pair of angle-action variables x_λ, y_λ and a single frequency ω_λ for simplicity. The phase space thus is $\mathbf{T}^\Lambda \times \mathbf{R}^\Lambda$, where $\mathbf{T}$ denotes the standard 1-torus, and the unperturbed Hamiltonian is

$$N = e + \sum_{\lambda \in \Lambda} \omega_\lambda y_\lambda .$$

The equations of motion are $\dot{x} = \omega$, $\dot{y} = 0$ in usual vector notation.—The frequencies ω are regarded as *parameters* varying freely over some open domain $\mathcal{O}$ in the parameter space $\mathbf{R}^\Lambda$ endowed with the sup-norm. The dependence of everything on these parameters, however, is not indicated for simplicity and brevity.

I consider perturbations of N having some sort of "spatial structure". Precisely, I consider Hamiltonians

$$H = N + P, \qquad P = \sum_{A \in \mathcal{S}} P_A(x_A, y_A),$$

where $\mathcal{S}$ is a collection of finite subsets of Λ —called *spatial strucure*—such that

$$A, B \in \mathcal{S},\ A \cap B \neq \emptyset \quad \Rightarrow \quad A \cup B \in \mathcal{S},$$

and the P_A "live on A". That is, P_A only depends on the configurations of the oscillators in A. The important feature of spatial structures is that they are preserved under Poisson brackets.

All quantitative aspects of the perturbation theory are expressed in terms of a nonnegative *weight function* denoted by $[A]$ such that

$$A \subseteq B \quad \Rightarrow \quad [A] \leq [B],$$
$$A \cap B \neq \emptyset \quad \Rightarrow \quad [A \cup B] + [A \cap B] \leq [A] + [B],$$

for all $A, B \in \mathcal{S}$. Given such a weight, let $w \geq 0$, and introduce the norm

$$|y|_w = \sum_{\lambda \in \Lambda} |y_\lambda| \, e^{w\langle\lambda\rangle}, \qquad \langle\lambda\rangle = \min_{\lambda \in A \in \mathcal{S} \cap \mathcal{S}} [A].$$

I then consider perturbations P that are real analytic on a complex neighbourhood $|\mathrm{Im}\, x|_\infty < r$, $|y|_w < s$ of the torus $\mathrm{T}^\Lambda \times \{0\}$ and are bounded with respect to the norm

$$|||P|||_{m,r,s} = \sum_{A \in \mathcal{S}} \|P_A\|_{r,s} e^{m[A]}.$$

Here,

$$\|P_A\|_{r,s} = \sum_{k \in \mathbf{Z}^A} |P_{A,k}|_s \, e^{r|k|}$$

is a weighted Fourier series norm as in Vittot (1985), and $|\cdot|_s$ denotes the sup over $|y|_w < s$. This norm indicates the key idea to treat spatial and Fourier expansion on exactly the same footing.

A vital rôle in any such perturbation theory is played by the small divisor conditions. Here they take the form

$$|\langle k, \omega\rangle| \geq \frac{\alpha}{\Delta(|k|)\Delta([\![k]\!])}, \qquad 0 \neq k \in \mathrm{Z}^\Lambda,$$

where α is a positive parameter, Δ some fixed normalized *approximation function* as introduced by Rüssmann (1980), and

$$|k| = \sum_{\lambda \in \Lambda} |k_\lambda|, \qquad [\![k]\!] = \min_{\mathrm{supp}\, k \subseteq A \in \mathcal{S}} [A],$$

where $\mathrm{supp}\, k = \{\lambda : k_\lambda \neq 0\}$. The precise characterization of approximation functions is not important here, but a useful example is $\Delta(t) = \exp(t/\log^\beta t)$ with $\beta > 1$. Note that the right hand side of the small divisor condition is zero when $\mathrm{supp}\, k$ is not finite, so it is not necessary to explicitly impose that restriction.

Clearly, the "heavier" the weight function $[\cdot]$ the likelier is the existence of strongly nonresonant frequencies ω. The objective therefore is to find spatial structures with not too heavy weights such that the set $\mathcal{O}_\alpha$ of all "good" frequencies in $\mathcal{O}$ is not empty for some small α and the resulting model is still interesting physically.

2. THE THEOREMS

For simplicity and convenience the following theorem refers to integer lattices only.

Theorem A. *Suppose $\Lambda \subseteq \mathbf{Z}^d$, and every set in $\mathcal{S}$ is connected. If there exists a $\sigma > 1$ such that*

$$[A] \geq |A| \log^\sigma |A|, \qquad \max_{i \in A} |i| \leq \exp([A] / \log^\sigma [A])$$

for all A with $|A| = \operatorname{card} A$ respectively $[A]$ sufficiently large, then there exists a probability measure μ on the parameter space $\mathbf{R}^\Lambda$ with support at any prescribed point such that

$$\mu(\mathbf{R}^\Lambda - \mathbf{R}^\Lambda_\alpha) = O(\alpha).$$

If the assumption of connectedness is dropped, then the first exponent has to be replaced by $1+\sigma$.

It follows that under the hypotheses of Theorem A the set $\mathcal{O}_\alpha$ is not empty for sufficiently small α whenever the set $\mathcal{O}$ has an interior point.—In the next theorem Ψ denotes an unbounded decreasing function on the positive real axis that is entirely defined in terms of the approximation function Δ.

Theorem B. *Suppose P is real analytic on $|\operatorname{Im} x|_\infty < r$, $|y|_w < s$ with $w \geq 0$. If*

$$s^{-1} |||P|||_{m,r,s} \leq \frac{\varepsilon_0 \alpha}{\Psi(\rho)\Psi(m-w)}$$

for some $0 < \rho < r/2$ and some $m > w$, where ε_0 is an absolute positive constant, then the Hamiltonian $H = N + P$ has an infinite dimensional invariant torus with a vectorfield conjugate to ω for every ω in $\mathcal{O}_\alpha$. These tori are close to the torus $y = 0$ with respect to the norm $|\cdot|_w$.

As usual, the constant ε_0 is very small—for example, $\varepsilon_0 = 2^{-39}$.

3. EXAMPLE: NEAREST NEIGHBOUR COUPLING

Let $\Lambda = \mathbf{Z}^d$ with $d \geq 1$, and let $\mathcal{S}$ be the smallest spatial structure containing the nearest neighbour sets $A_i = \{ j : |j - i| \leq 1 \}$, with $|\cdot| = |\cdot|_\infty$. Consider the Hamiltonian

$$H = \sum_i \omega_i y_i + \varepsilon \sum_i P_{A_i}(x, y),$$

assuming that

$$P_{A_i} = O(|y_{A_i}|^\lambda)$$

uniformly in i for some exponent $\lambda > 1$.

Let $[\cdot]$ be any weight function satisfying the hypothesis of Theorem A. Choose $w = 1$. Picking any initial condition y^o with $|y^o|_1 = 2$ and expanding H in a ball of radius 1 around it, we have

$$\|P_{A_i}\|_{r,1} \leq C \max_{j \in A_i} e^{-\lambda[j]}$$

for some $r > 0$ and thus

$$|||P|||_{m,r,1} = \sum_i \|P_{A_i}\|_{r,1} e^{m[A_i]} \leq C \sum_{i, j \in A_i} e^{m[A_i] - \lambda[j]}.$$

Now, if

$$\lambda > \lambda_* = \operatorname*{ess\,sup}_{j \in A_i} \frac{[A_i]}{[j]} \geq 1,$$

then the latter sum is finite for some $m > 1$, and so for sufficiently small ε there exist infinite dimensional tori localized like $y_i^o \sim e^{-[i]}$.

Here are now some specific examples. Choosing

$$[A] = \max_{i \in A} |i|^{d+\delta},$$

where d is the dimension of the lattice and $\delta > 0$, one obtains $\lambda_* = 1$ and $y_i^o \sim e^{-|i|^{d+\delta}}$. This is the example of Fröhlich, Spencer, Wayne (1986). But choosing

$$[A] = \sum_{i \in A} |i|,$$

one finds that $\lambda_* = 2d + 1$ and $y_i^o \sim e^{-|i|}$. Hence the decay is just exponential at the expense of increasing the exponent λ.

This, however, can be improved on by choosing

$$[A] = \sum_{|i-A| \leq t} |i|,$$

with some $t \geq 0$. As the "thickness" t increases, $\lambda_* \downarrow 1$ and $y_i^o \sim e^{-c|i|}$ with $c \uparrow \infty$. The same is true for

$$[A] = \sum_{|i-A| \leq t} \log^\sigma |i|, \qquad \sigma > 1.$$

Now $y_i^o \sim e^{-c \log^\sigma |i|}$. For a somewhat related result see Vittot (1985).

Of course there is nothing special about nearest neighbour sets, and everything works the same when starting with neighbouring sets A_i of any uniform size, as it was done by Vittot and Bellissard (1985). Also of interest are *hierarchical* structures, where $A \cap B \neq \emptyset$ implies $A \subseteq B$ or $B \subseteq A$. The theory outlined here is also useful to improve results on finite chains of oscillators as considered by Wayne (1984) and Vittot (1985). Finally, the classical analytic KAM-theorem is recovered by letting $\Lambda = \{1, \dots, N\}$ and $\mathcal{S} = \{\Lambda\}$ with $[\Lambda] = 0$.

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