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The Hausdorff Dimension of Small Divisors for Lower Dimensional KAM-Tori

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1 The Problem and the Results

Let Ω be a bounded domain in $\mathbb{R}^m$,

$$\phi : \Omega \to \mathbb{R}^n$$

a Lipschitz continuous map, and $\psi : (0, \infty) \to (0, \infty)$ a function converging monotonically to zero on the integers at infinity. We are interested in the metric properties of the set $\mathfrak{W}_h(\psi)$ of those points ω in Ω for which the inequality

$$|k \cdot \omega - l \cdot \phi(\omega)| < \psi(|k| + |l|)$$

has infinitely many distinct integer solutions (k, l) in $\mathbb{Z}^m \times \mathbb{Z}^n$ satisfying

$$|l| \leq h,$$

h a fixed nonnegative integer. Here,

$$k \cdot \omega = \sum_{i=1}^{m} k_i \omega_i, \qquad |k| = \sum_{i=1}^{m} |k_i|,$$

and so on. Thus, $\mathfrak{W}_h(\psi)$ consists of those m-vectors ω in Ω, whose components together with the components of the dependent n-vector $\phi(\omega)$ are well approximable with respect to ψ by a certain class of $m + n$-vectors.

To describe these sets more geometrically and succinctly, let

$$M = \{ (\omega, -\phi(\omega)) : \omega \in \Omega \}$$

be the graph of $-\phi$. This is an m-dimensional Lipschitz manifold embedded in $\mathbb{R}^{m+n}$ which is lipeomorphic to Ω via the projection P which maps $m+n$-vectors onto their first m components. The inverse image of $\mathfrak{W}_h(\psi)$ under P in M is the set

$$W_h(\psi) = \{ x \in M : |q \cdot x| < \psi(|q|) \text{ i.o. in } I_h \},$$

where 'i.o.' means 'infinitely often' and I_h is the 'cylinder' given by

$$I_h = \{ q \in \mathbb{Z}^{m+n} : \sum_{i=1}^{n} |q_{m+i}| \le h \} = \mathbb{Z}^m \times \mathbb{Z}_h^n,$$

where $\mathbb{Z}_h^n = \{ l \in \mathbb{Z}^n : |l| \le h \}$. Hence $W_h(\psi)$ consists of those points on the manifold M which are well approximable relative to ψ by integer vectors that lie in the set I_h. Conversely,

$$\mathfrak{W}_h(\psi) = P W_h(\psi).$$

Thus, there is no difference between $\mathfrak{W}_h(\psi)$ and $W_h(\psi)$ as far as metric properties are concerned. The latter set, however, is somewhat more convenient to use conceptually and notationally.

These sets $W_h(\psi)$ 'interpolate' between two more familiar types of sets arising in the metric theory of Diophantine approximations of independent and dependent quantities respectively, in the terminology of Sprindžuk (1979). In the 'independent' case there are no dependent quantities $\phi(\omega)$ at all, corresponding to

$$h = 0, \qquad I_0 = \mathbb{Z}^m \times \{0\}.$$

So one considers the set $W_0(\psi) = \{ x \in M : |q \cdot x| < \psi(|q|) \text{ i.o. in } \mathbb{Z}^m \times \{0\} \}$, or rather its projection

$$\mathfrak{W}_0(\psi) = \{ \omega \in \Omega : |k \cdot \omega| < \psi(|k|) \text{ i.o. in } \mathbb{Z}^m \}$$

of 'almost rationally dependent points' in Ω relative to ψ. In the 'dependent' case, the points x are confined to the manifold M, but all possible approximations by integer vectors from the ambient space are allowed. That is,

$$h = \infty, \qquad I_\infty = \mathbb{Z}^m \times \mathbb{Z}^n,$$

and

$$W_\infty(\psi) = \{ x \in M : |q \cdot x| < \psi(|q|) \text{ i.o. in } \mathbb{Z}^{m+n} \}.$$

Obviously, we have

$$W_0(\psi) \subseteq W_h(\psi) \subseteq W_\infty(\psi)$$

for all h with $0 < h < \infty$.

The sets $W_h(\psi)$ with $h = \infty$ studied in the theory of Diophantine approximations on manifolds are difficult to analyse. Usually, further restrictions on the manifold M are required to determine their Hausdorff dimension such as an upper bound on the codimension n or the nonvanishing of some sectional curvatures (Sprindžuk 1979, Dodson, Rynne and Vickers 1990). In contrast, the case $0 < h < \infty$ turns out to be not very different from the case $h = 0$, where no such restrictions arise. The only extra assumption needed here is the Lipschitz property of ϕ.

To state this result conveniently, let us introduce the following notion. A function ψ *has finite m-volume*, if ψ maps the positive real axis monotonically into itself and satisfies

$$\sum_{r=1}^{\infty} r^{m-1} \psi(r) < \infty.$$

In particular, ψ has to converge to zero on the integers at infinity.

Theorem 1. *Suppose $\phi : \Omega \to \mathbb{R}^n$ is Lipschitz and ψ has finite $m-1$-volume with $m \ge 2$ and $n \ge 1$. Then $W_h(\psi)$ has measure zero and Hausdorff dimension*

$$\dim W_h(\psi) = m - 1 + \frac{m}{\lambda + 1}$$

for every h in $0 \le h < \infty$, where

$$\lambda = \liminf_{r \to \infty} -\frac{\log \psi(r)}{\log r} \ge m - 1$$

denotes the lower order of $1/\psi$ at infinity.

An analogous result holds when in the definition of the W-sets the absolute value $|q \cdot x|$, which is the distance of $q \cdot x$ from zero, is replaced by its distance to the integers, $\|q \cdot x\| = \min_{k \in \mathbb{Z}} |q \cdot x - k|$. In this case, ψ has to be assumed to have finite m-volume with $m \ge 1$, and in the result the term $m/(\lambda + 1)$ has to be replaced by $(m+1)/(\lambda+1)$. This requires only minor modifications of the proofs in Section 4 which we forego.

These sets $W_h(\psi)$ are closely connected with *exceptional sets*

$$E_h(\psi) = \left\{ x \in M : \inf_{0\neq q\in I_h} |q\cdot x| / \psi(|q|) = 0 \right\}$$

of small divisors arising in the KAM-theory of m-dimensional elliptic invariant tori in Hamiltonian systems of $m+n$ degrees of freedom. The sets $E_h(\psi)$ obviously contain the set

$$R_h = \left\{ x \in M : |q\cdot x| = 0 \text{ for some } 0 \neq q \in I_h \right\}$$

of *exact resonances*. An exact resonance, however, need not belong to $W_h(\psi)$, since the index set I_h is not a module, when $h \neq 0$. So exceptional points need not be well approximable. Rather, the following purely set theoretic identity holds.

E-Identity. *$E_h(\psi) = X_h(\psi) \cup R_h$, where $X_h(\psi) = \bigcap_{\alpha>0} W_h(\alpha\psi)$.*

As an almost immediate consequence of this identity and Theorem 1 we obtain

Theorem 2. *Suppose $\phi : \Omega \to \mathbb{R}^n$ is Lipschitz and ψ has finite $m-1$-volume with $m \geq 2$ and $n \geq 1$. Then, for $0 \leq h < \infty$, the set $E_h(\psi)$ has measure zero, if R_h has measure zero, and*

$$\dim E_h(\psi) = \dim W_h(\psi),$$

if $\dim R_h \leq m-1$.

These last two results are also modified as described above, when again $|q\cdot x|$ is replaced by $\|q\cdot x\|$ in the definitions of $E_h(\psi)$ and R_h.

So far, our setting has been finite dimensional. But interestingly, none of our results depends in any way on the codimension n of M. Therefore the question arises whether they extend in any way to the case $n = \infty$. Such extensions are relevant for perturbation theories of finite dimensional invariant tori in *infinite* dimensional Hamiltonian systems such as nonlinear wave or Schrödinger equations, which were investigated independently by Kuksin (1988) and Wayne (1990).

Thus, we now consider a map

$$\phi : \Omega \to \mathbb{R}^\infty,$$

which we assume to be Lipschitz with respect to the sup-norm on the target space. We continue to write M for its graph and $W_h(\psi), \ldots$ for its subsets under consideration, but understand their Lebesgue measure and Hausdorff dimension to refer to their projections onto Ω. Thus, for example, we understand $|W_h(\psi)|$ to denote the m-dimensional Lebesgue measure $|PW_h(\psi)|$, and similarly for its Hausdorff dimension.

There are now two quite distinct cases, one of which is an immediate extension of the finite dimensional case, and which is characterized by the following *finiteness condition*: *for every $x \in M$,*

$$\operatorname{card}\left\{ q \in I_h : |q| \leq s,\ |q\cdot x| \leq t \right\} < \infty$$

for every finite $s, t > 0$. This condition is trivially satisfied in the finite dimensional case, so the following theorem includes and extends our first two results.

Theorem 3. *Suppose $\phi : \Omega \to \mathbb{R}^n$ is Lipschitz with respect to the sup-norm and ψ has finite $m-1$-volume with $2 \leq m < \infty$ and $1 \leq n \leq \infty$. Moreover, suppose the finiteness condition is satisfied. Then, for $0 \leq h < \infty$, the set $W_h(\psi)$ has measure zero and Hausdorff dimension*

$$\dim W_h(\psi) = m - 1 + \frac{m}{\lambda+1}$$

for $\lambda \geq m-1$, where λ denotes the lower order of $1/\psi$ at infinity. Moreover, the set $E_h(\psi)$ has measure zero, if R_h has measure zero, and $\dim E_h(\psi) = \dim W_h(\psi)$, *if* $\dim R_h \leq m-1$.

The finiteness condition is indeed necessary. Otherwise, $W_h(\psi)$ may happen to have interior points and hence positive measure and full dimension. An example is given in the next section.

Nonetheless, in such cases we can still say something about the set $E_h(\psi)$, which is ‘smaller’ than $W_h(\psi)$. To this end, we consider the set $A_h(\phi)$ of all accumulation points of the points $l\cdot\phi(\omega)$ with $|l| \leq h$ and $\omega \in \Omega$. More precisely, $a \in A_h(\phi)$, iff there exists a sequence of integer vectors l_i with $|l_i| \leq h$ and a point $\omega \in \Omega$ such that $l_i\cdot\phi(\omega) \to a$ as $i \to \infty$.

The next theorem includes and extends Theorem 2.

Theorem 4. *Suppose* $\phi\colon \Omega \to \mathbb{R}^n$ *is Lipschitz with respect to the sup-norm and* ψ *is decreasing and has finite* $m-1$*-volume with* $2 \le m < \infty$ *and* $1 \le n \le \infty$*. Moreover, suppose that, for fixed* $0 \le h < \infty$*, the set* $A_h(\phi)$ *is countable and does not contain 0, and that* $\dim R_h \le m-1$*. Then the set* $E_h(\psi)$ *has measure zero and Hausdorff dimension*

$$\dim E_h(\psi) = m - 1 + \frac{m}{\lambda+1},$$

where $\lambda \ge m-1$*.*

Unlike the finite dimensional case, however, there are *no* analogous results in infinite dimensions, when $\|q\cdot x\|$ is substituted for $|q \cdot x|$. Except for very special situations the corresponding sets will then have interior points and even be equal to M. We illustrate this point in the next section.

2 An Example

To illustrate our results we consider a map $\phi\colon \Omega \to \mathbb{R}^\infty$ whose component maps ϕ_i, $i \ge 1$, are of the form

$$\phi_i(\omega) = \mu_i + \eta_i(\omega)$$

with

$$\mu_i = i^d, \qquad |\eta_i|_\Omega = o(i^{d-1})$$

for some real exponent $d > 0$. Here, $|\cdot|_\Omega$ denotes the sup-norm over the open domain $\Omega \subset \mathbb{R}^m$, and the o-notation means that $|\eta_i|_\Omega / i^{d-1} \to 0$ as $i \to \infty$. We also assume that

$$\sup_{i\ge1} |\eta_i|_\Omega < \infty,$$

so that ϕ is Lipschitz with respect to the sup-norm.

Mappings of this kind are given, for example, by the sequence of eigenvalues of some unbounded self adjoint operators with pure point spectrum depending on parameters. A well known member of this class is the Sturm-Liouville operator $-d^2/dx^2 + q$ with Dirichlet boundary conditions on the real interval $[0,\pi]$ depending on a potential $q \in L^2[0,\pi]$. Its spectrum is given by a strictly increasing sequence of real numbers

$$\phi_i(q) = i^2 + \frac{1}{\pi}\int_0^\pi q(x)\,dx + \eta_i(q), \qquad i \ge 1,$$

which are real analytic in q and satisfy $\sum_{i\ge1} |\eta_i|^2 < \infty$ uniformly on bounded subsets of $L^2[0,\pi]$ (Pöschel and Trubowitz 1987).

We confine ourselves to the case

$$h = 2 \iff |l| \le 2$$

relevant for applications in KAM-theory (Kuksin 1988, Wayne 1990) and drop the index h from the set notation in the sequel. Then there are basically two different types of expressions $q\cdot x = k\cdot\omega - l\cdot\phi(\omega)$, depending on whether the nonzero coordinates of $l \ne 0$ have the same or opposite sign. The first case is harmless, because the collection of points $l\cdot\phi(\omega)$ for all such l is *discrete* for each ω. It is the second case which makes all the difference.

The contrast between these two cases is analogous to the difference between the (complementary) Poincaré and Siegel domains (Arnold 1983). The Poincaré domain consists of n-tuples $(\lambda_1,\dots,\lambda_n)$ of complex eigenvalues, for which the convex hull of the points $\lambda_1,\dots,\lambda_n$ in the complex plane does not contain the origin. As a result, the resonant sets are distributed discretely in the Poincaré domain. The Siegel domain is the complement of the Poincaré domain, so that the origin lies in the convex hull of the set of eigenvalues; in this case both resonant and nonresonant sets are dense.

Case A. *If* $d > 1$*, then the finiteness condition is satisfied and so Theorem 3 applies.*

Proof. Fix ω in Ω and $s, t > 0$ and consider the set of integer vectors (k,l) satisfying $|k| + |l| \le s$ and $|k\cdot\omega - l\cdot\phi(\omega)| \le t$. Then of course $|k| \le s$, which restricts k to a finite set and moreover implies

$$|l\cdot\phi(\omega)| \le s\,|\omega| + t.$$

Among those l with coordinates of the same sign only finitely many can satisfy this inequality because of the asymptotic behaviour of $\phi(\omega)$. And for $l = e_i - e_j$, $i \ne j$, we have

$$\begin{aligned} |l\cdot\phi(\omega)| &= \left|\phi_i(\omega) - \phi_j(\omega)\right| \\ &\ge \left|i^d - j^d\right| - |\eta_i(\omega)| - \left|\eta_j(\omega)\right| \\ &\ge \frac{1}{2}(i^{d-1} + j^{d-1}) - o(i^{d-1}) - o(j^{d-1}) \end{aligned}$$

which again restricts i and j and hence l to a finite set. ∎

For $d = 1$, the finiteness condition is no longer satisfied. For example, for any integer $n \neq 0$ there are infinitely many integer vectors

$$l_i^n = e_{i+n} - e_i, \qquad i \geq \max(1, 1-n),$$

such that

$$\begin{aligned} l_i^n \cdot \phi(\omega) &= \phi_{i+n}(\omega) - \phi_i(\omega) \\ &= n + \eta_{i+n}(\omega) - \eta_i(\omega) \\ &\to n \qquad \text{as } i \to \infty \end{aligned}$$

for every $\omega \in \Omega$. Even more, the conclusion of Theorem 3 no longer holds.

Case B. *If $d = 1$, then $PW(\psi)$ contains the nonempty, open set*

$$\bigcup_k \bigcup_{n \neq 0} \{ \omega \in \Omega : |k \cdot \omega - n| < \psi(|k| + 2) \}.$$

Proof. The set in question is clearly open and not empty, since Ω is assumed to be open and thus contains exact resonances ω satisfying $k \cdot \omega = n$ for some k and $n \neq 0$.

Now, given any ω in this set with appropriate k and n, we have

$$\left| k \cdot \omega - l_i^n \cdot \phi(\omega) \right| \to |k \cdot \omega - n| \qquad \text{as } i \to \infty$$

by the observation made above. Hence,

$$\left| k \cdot \omega - l_i^n \cdot \phi(\omega) \right| < \psi(|k| + 2)$$

for infinitely many i, and consequently $\omega \in PW(\psi)$ by definition. ∎

Nonetheless we can still apply Theorem 4, if we make a further, natural assumption.

Case C. *If $d = 1$, and if the sets $Q_{kl} = \left\{ \omega \in \Omega : k \cdot \omega - l \cdot \phi(\omega) = 0 \right\}$ have codimension greater or equal one for every $(k, l) \neq 0$, $|l| \leq 2$, then Theorem 4 applies.*

Proof. By the asymptotics of ϕ, finite accumulation points of the collection $l \cdot \phi(\omega)$, $|l| \leq 2$, are only obtained from integer vectors $l = e_i - e_j$, $i \neq j$, for which we have

$$l \cdot \phi(\omega) = i - j + o(1)$$

independently of ω. It follows that

$$A(\phi) = \mathbb{Z} \backslash \{0\},$$

which is a countable set not containing 0 as required. Moreover, since $PR = \bigcup_{(k,l) \neq 0} Q_{kl}$, we also have $\dim R \leq \sup \dim Q_{kl} \leq m - 1$ by hypothesis. ∎

The proof of the Auxiliary Lemma in the following section — where also the Lipschitz-semi-norm $|\phi|_L$ is defined — shows that every nonempty set Q_{kl} with $|k| \geq 4\,|\phi|_L + 1$ is a Lipschitz manifold of codimension 1. Thus only finitely many k actually need to be considered. Also, in many cases the same is true for the l's, so that all in all only finitely many sets Q_{kl} have to be inspected.

For $d < 1$, however, everything is lost. Since in this case $(i+n)^d - i^d \to 0$ as $i \to \infty$, we have

$$l_i^n \cdot \phi(\omega) \to 0 \qquad \text{as } i \to \infty$$

for any $n \neq 0$ and every $\omega \in \Omega$. Thus, every ω satisfies the criterion for belonging to $PW(\psi)$ and $PE(\psi)$ simply by letting $k = 0$. Thus we have

Case D. *If $d < 1$, then $W(\psi) = M$ and $E(\psi) = M$.*

Finally, we look at what happens when in the definition of all the sets the absolute value of $q \cdot x$ is replaced by its distance to the integers. Clearly, the finiteness condition becomes meaningless, since $\|q \cdot x\| \leq \frac{1}{2}$ for every q and every x. In fact, there is nothing we can do and all the points on M are well approximable with respect to ψ.

Case E. *If the distance-to-integers is used, then $W(\psi) = M$ and $E(\psi) = M$ for every exponent d.*

Proof. Take any $\omega \in \Omega$. Then the numbers $\|\phi_i(\omega)\|$, $i \geq 1$, have some accumulation point in the interval $[0, \frac{1}{2}]$ by compactness. Choosing two different subsequences with the same accumulation point we find that the numbers $\left\| \phi_i(\omega) - \phi_j(\omega) \right\|$, $i \neq j$, accumulate at 0. Thus, there exists a sequence of distinct vectors l_i, $|l_i| = 2$, such that

$$\|l_i \cdot \phi(\omega)\| = \|0 \cdot \omega - l_i \cdot \phi(\omega)\| \to 0 \qquad \text{as } i \to \infty .$$

Again, this means that ω satisfies the criterion for belonging both to $PW(\psi)$ and $PE(\psi)$. ∎

3 An Auxiliary Lemma

We begin by studying the effect of a single dependent quantity represented by a real valued function u on Ω. We assume this function to be *uniformly* Lipschitz, thus having a finite Lipschitz constant

$$|u|_L = \sup_{\omega \neq \omega'} \frac{\left|u(\omega) - u(\omega')\right|}{|\omega - \omega'|}$$

on Ω. The norm in the denominator is understood as the sup-norm.

Auxiliary Lemma. *Suppose $\Omega \subset \mathbb{R}^m$ is bounded and measurable, and $u : \Omega \to \mathbb{R}$ is uniformly Lipschitz. If $|k| \geq 2\,|u|_L + 1$, then the set*

$$D = \left\{ \omega \in \Omega : |k \cdot \omega - u(\omega)| \leq \delta \right\}$$

has Lebesgue measure

$$|D| \leq C\delta / |k|$$

with $C = 4d^{m-1}\sqrt{m}$, where d denotes the exterior diameter of Ω with respect to the sup-norm. Moreover, D has a covering by at most $cs^{-(m-1)}$ cubes of side length $s = \delta / |k|$, where c depends only on the dimension m and the diameter d.

Proof. First, we extend u to a function U on all of $\mathbb{R}^m$ preserving its Lipschitz constant $\lambda = |u|_L$. For $z \in \mathbb{R}^m$ set

$$U(z) = \sup_{\omega \in \Omega} \big(u(\omega) - \lambda\,|z - \omega| \big).$$

We claim that

$$U \mid \Omega = u, \qquad |U|_L = |u|_L ,$$

where $|U|_L$ denotes the Lipschitz constant of U on the whole of euclidean space.

Indeed, by the triangle inequality,

$$\big(u(\omega) - \lambda \left|z' - \omega\right| \big) \geq \big(u(\omega) - \lambda\,|z - \omega| \big) - \lambda \left|z - z'\right| .$$

Taking suprema over ω we obtain $U(z') \geq U(z) - \lambda \left|z - z'\right|$, or $U(z) - U(z') \leq \lambda \left|z - z'\right|$. Interchanging z and z' we obtain

$$\left|U(z) - U(z')\right| \leq \lambda \left|z - z'\right| .$$

The rest of the argument is obvious, thus proving our claim.

We may now write $D = Z \cap \Omega$ with

$$Z = \left\{ z \in \mathbb{R}^m : |k \cdot z - U(z)| \leq \delta \right\}$$

and focus attention on Z.

Choose $v \in \{-2, 2\}^m$ in such a way that $k \cdot v = 2\,|k|$ and take w in the subspace $v^{\perp}$ orthogonal to v. Consider the Lipschitz continuous function

$$\Phi(z) = k \cdot z - U(z)$$

along the line

$$z = w + tv, \qquad -\infty < t < \infty.$$

The slope of $k \cdot z$ as a function of t is exactly $2\,|k|$, whereas the Lipschitz constant of $U(z)$ as a function of t is bounded by 2λ. It follows that $\Phi(w + tv)$ is monotonically increasing in t with minimal slope $2\,|k| - 2\lambda \geq |k| \geq 1$ by hypotheses. Hence, this function takes values in the real interval $[-\delta, \delta]$ on a unique t-interval $[a, b]$ with length

$$b - a \leq 2\delta / |k| .$$

This bound is independent of w. Moreover, as functions of w, the endpoints a and b are Lipschitz with

$$|a|_L , |b|_L \leq \frac{1}{2} \cdot \frac{|k| + \lambda}{|k| - \lambda}.$$

For example, when $\Phi(w + av)$ and $\Phi(w' + a'v)$ are both equal to $-\delta$, then

$$\begin{aligned} 2(|k| - \lambda) \left|a - a'\right| &\leq \left|\Phi(w + av) - \Phi(w + a'v)\right| \\ &= \left|\Phi(w' + a'v) - \Phi(w + a'v)\right| \\ &\leq (|k| + \lambda) \left|w - w'\right| , \end{aligned}$$

which proves our estimate.

Thus we have $D = Z \cap \Omega$, where

$$Z = \left\{ z = w + tv : a(w) \le t \le b(w),\ w \in v^{\perp} \right\}$$

is a closed set bounded by the graphs of two uniformly Lipschitz functions a and b over $v^{\perp}$, which are at most $4\sqrt{m} \cdot \delta / |k|$ apart with respect to the euclidean distance. From this description of D the statements easily follow. ∎

4 The Proofs in the Finite Dimensional Case

From now on we will drop the subscript h from our set notation for simplicity, since none of our arguments and constructions depends on its specific value.

We will assume the map $\phi : \Omega \to \mathbb{R}^n$ to be *uniformly* Lipschitz both in the finite and infinite dimensional case, thus having a finite Lipschitz constant

$$|u|_L = \sup_{\omega \neq \omega'} \frac{\left|\phi(\omega) - \phi(\omega')\right|}{|\omega - \omega'|},$$

where both norms are understood as sup-norms. This causes no loss of generality. For, we may always cover Ω by a countable number of subsets with a finite Lipschitz constant, and proving our claims for each such subset will then prove them also for their countable union. Hence we may as well assume ϕ to be uniformly Lipschitz on all of Ω.

First we prove the E-identity,

$$E(\psi) = X(\psi) \cup R, \quad \text{where} \quad X(\psi) = \bigcap_{\alpha > 0} W(\alpha\psi).$$

As we pointed out, this identity is of a purely set theoretic nature. It holds for any ϕ and any ψ and in infinite codimensions as well.

Proof of the E-identity. To prove the left inclusion, suppose that $x \in E(\psi)$ and $x \notin R$. Then $|q \cdot x| \neq 0$ for all $0 \neq q \in I$. By induction, we may then choose $1 > \alpha_1 > \alpha_2 > \cdots \to 0$ and q_i, $i \ge 1$, in I such that

$$\alpha_{i+1}\psi(|q_i|) < |q_i \cdot x| < \alpha_i \psi(|q_i|)$$

for all $i \ge 1$. These q_i are thus all different from each other and satisfy

$$|q_i \cdot x| < \alpha_j \psi(|q_i|), \qquad i \ge j \ge 1,$$

whence $x \in W(\alpha_j \psi)$ for all $j \ge 1$ and consequently $x \in X(\psi)$.

To prove the right inclusion suppose that $x \in X(\psi)$ and $x \notin R$. Then we may construct exactly the same sequences of α_i and q_i as above, this time concluding that

$$|q_i \cdot x| / \psi(|q_i|) < \alpha_i \to 0 \quad \text{as} \quad i \to \infty,$$

whence $x \in E(\psi)$. Since R is obviously contained in $E(\psi)$, we are done. ∎

Now let

$$V_q(\psi) = \left\{ x \in M : |q \cdot x| < \psi(|q|) \right\}$$

for $q \in I$. Furthermore, let $m(\lambda) = m - 1 + \dfrac{m}{\lambda + 1}$.

Lemma 1. *Suppose $\phi : \Omega \to \mathbb{R}^n$ is uniformly Lipschitz and ψ is $m-1$-summable. If*

$$Y \subseteq \bigcup_{|q| \ge N} V_q(\psi)$$

for every $N \ge 1$, then $|Y| = 0$ and $\dim Y \le m(\lambda)$.

In particular, the 'lim-sup' set $\bigcap_N \bigcup_{|q| \ge N} V_q(\psi)$ itself has measure zero and Hausdorff dimension not bigger than $m(\lambda)$.

Proof. For $q = (k, l)$, the projection of $V_q(\psi)$ into Ω is the set

$$PV_q(\psi) = \left\{ \omega \in \Omega : |k \cdot \omega - l \cdot \phi(\omega)| < \psi(|q|) \right\}.$$

When $|q| \ge N$ is sufficiently large, the Auxiliary Lemma applies to these sets with $\delta = \psi(|q|)$ and $u = l \cdot \phi$, since $|u|_L \le |l|\,|\phi|_L \le h\,|\phi|_L$ is uniformly bounded and $|k| \ge |q| - |l| \ge N - h$ is large. Hence,

$$\left|V_q(\psi)\right| \ll \psi(|q|) / |q|$$

and so, for all large N,

$$|Y| \le \sum_{|q| \ge N} \left|V_q(\psi)\right| \ll \sum_{|q| \ge N} \psi(|q|) / |q| \ll \sum_{r \ge N} r^{m-2} \psi(r),$$

where $\ll$ indicates inequality with a positive factor. The right hand side tends to zero as N tends to infinity, since ψ is assumed to have finite $m-1$-volume. So Y has measure zero. In particular, Y is measurable.

To obtain the upper bound for its Hausdorff dimension, let $\varepsilon > 0$ be small and $t = m(\lambda - \varepsilon) + \varepsilon$, that is,

$$t = m - 1 + \frac{m}{\lambda + 1 - \varepsilon} + \varepsilon > m(\lambda).$$

By the Auxiliary Lemma, the set $PV_q(\psi)$ has a covering by at most $cs_q^{-(m-1)}$ cubes of side length $s_q = \psi(|q|)/|q|$. The collection $\mathcal{C}_N$ of all those covers with $|q| \geq N$ is a covering of PY with t-volume

$$\begin{aligned} V^t(\mathcal{C}_N) &\ll \sum_{|q|\geq N} s_q^{-(m-1)} s_q^t \\ &\ll \sum_{|q|\geq N} \left(\frac{\psi(|q|)}{|q|} \right)^{t-(m-1)} \\ &\ll \sum_{r\geq N} r^{m-1} \left(\frac{\psi(r)}{r} \right)^{t-(m-1)} . \end{aligned}$$

Now, for all N sufficiently large,

$$-\frac{\log \psi(r)}{\log r} \geq \lambda - \varepsilon \quad \text{for} \quad r \geq N$$

by the definition of λ. Thus, $\psi(r) \leq r^{-(\lambda-\varepsilon)}$ for $r \geq N$, and so

$$V^t(\mathcal{C}_N) \ll \sum_{r\geq N} r^{m-1+(m-1-t)(\lambda+1-\varepsilon)} \ll \sum_{r\geq N} r^{-1-\varepsilon} \to 0$$

as N tends to infinity by the definition of t. This shows that $\dim Y \leq t = m(\lambda - \varepsilon) + \varepsilon$. This holds for all $\varepsilon > 0$, so $\dim Y \leq m(\lambda)$. ∎

Lemma 2.

$$W(\psi) \subseteq \bigcup_{|q|\geq N} V_q(\psi)$$

for every $N \geq 1$ *and so* $|W(\psi)| = 0$ *and* $\dim W(\psi) \leq m(\lambda)$.

Proof. If $x \in W(\psi)$, then there are infinitely many q_i in I with $|q_i \cdot x| < \psi(|q_i|)$, that is, $x \in V_{q_i}(\psi)$. Necessarily, in finite dimensions, $|q_i| \to \infty$ as $i \to \infty$. The inclusion thus follows, and so does the rest by Lemma 1. ∎

To obtain the corresponding lower bound for $\dim W(\psi)$ we resort to an extension of a result in Diophantine approximations concerning the set

$$W_0(\psi) = \left\{ x \in M : |q \cdot x| < \psi(|q|) \text{ i.o. in } \mathbb{Z}^m \times \{0\} \right\}$$

Its projection into Ω is the set

$$PW_0(\psi) = \left\{ \omega \in \Omega : |k \cdot \omega| < \psi(|k|) \text{ i.o. in } \mathbb{Z}^m \right\}$$

for which we have $\dim PW_0(\psi) = m(\lambda)$. This follows from a general theorem in Dodson et al (1990) a special case of which is reviewed in the appendix. Now we obviously have

Lemma 3.

$$W_0(\psi) \subseteq W(\psi) \cap E(\psi)$$

and so $m(\lambda) \leq \dim W_0(\psi) \leq \min\left\{\dim W(\psi), \dim E(\psi)\right\}$

These last two lemmata provide the proof of Theorem 1. Theorem 2 follows immediately, since

$$\begin{gathered} |E(\psi)| \leq |W(\psi)| + |R| \\ \dim W_0(\psi) \leq \dim E(\psi) \leq \max\left\{\dim W(\psi), \dim R\right\} \end{gathered}$$

by the E-identity, and $\dim R \leq \dim W(\psi)$ by assumption.

5 The Proofs in the Infinite Dimensional Case

The lower bounds for the Hausdorff dimensions of $W(\psi)$ and $E(\psi)$ are independent of any consideration of ϕ, since they rely on the inclusion $W_0(\psi) \subseteq W(\psi) \cap E(\psi)$ of Lemma 3. Hence, these lower bounds are also valid here without any further assumptions. It is only their upper bounds and Lebesgue measure that require further attention.

Assuming the finiteness hypotheses there is hardly anything to do.

Proof of Theorem 3. We show that the inclusion of Lemma 2,

$$W(\psi) \subseteq \bigcup_{|q|\geq N} V_q(\psi) \quad \text{for all large } N,$$

still holds. Then everything follows from Lemma 1 and the E-identity as before. Indeed, if $x \in W(\psi)$, then there are infinitely many q_i with $|q_i \cdot x| < \psi(|q_i|)$, that is, $x \in V_{q_i}(\psi)$. If we had $|q_i| \le s < \infty$ for all i, then

$$|q_i \cdot x| \le t = \max_{1 \le r \le s} \psi(r) < \infty$$

for all i, leading to a contradiction to our finiteness assumption. Hence $|q_i| \to \infty$, proving the inclusion. ∎

The proof of Theorem 4 is an extension of the preceding argument.

Proof of Theorem 4. By the E-identity, $E(\psi) = X(\psi) \cup R$. We show that for all $N \ge 1$,

$$X(\psi) \subseteq \bigcup_{|q| \ge N} V_q(\psi) \cup S,$$

where S is the set of all $x = (\omega, \phi(\omega))$ in M such that $k \cdot \omega \in A(\phi)$ for some $k \ne 0$. In other words,

$$PS = \bigcup_{0 \ne k \in \mathbb{Z}^m} \bigcup_{a \in A(\phi)} \{ \omega \in \Omega : k \cdot \omega = a \}.$$

For countable $A(\phi)$, this set is a countable union of subsets of Ω of dimension $\le m-1$, hence $|S| = 0$ and $\dim S \le m-1$. Thus, given this covering of $X(\psi)$ the theorem will again follow by Lemma 1.

So let $x \in X(\psi)$. Then, by its definition, there exist sequences α_i and q_i with

$$|q_i \cdot x| < \alpha_i \psi(|q_i|), \qquad \alpha_i \to 0.$$

If $|q_i| \to \infty$ for some subsequence, we are done. Otherwise, the $|q_i|$ are uniformly bounded by a finite number s, and we may extract a subsequence of q_i's such that $q_i = (k, l_i)$ with a fixed k for all i. Then writing $x = (\omega, \phi(\omega))$,

$$|q_i \cdot x| = |k \cdot \omega - l_i \cdot \phi(\omega)| < \alpha_i \sup_{1 \le r \le s} \psi(r) \to 0,$$

that is, the sequence $l_i \cdot \phi(\omega)$ converges to $k \cdot \omega$. But this implies that $k \cdot \omega = a \in A(\phi)$. In particular, $k \ne 0$, since we excluded 0 from $A(\phi)$. It follows that $\omega \in PS$ and $x \in S$. This establishes our covering of $X(\psi)$. ∎

The proof shows that it actually suffices to assume $A(\phi)$ to have Hausdorff dimension 0, instead of being countable, for Theorem 4 to be true.

Appendix: A lower bound for the Hausdorff dimension

Let $\mathcal{Z} = \{ Z_\alpha : \alpha \in I \}$ be a countable collection of affine subspaces Z_α of $\mathbb{R}^m$ of equal dimension d, $0 \le d < m$, and let $[\alpha]$ denote a positive weight assigned to each of the indices α. Given an open set $\Omega \in \mathbb{R}^m$ and a bounded function $\psi : (0,\infty) \to (0,\infty)$ converging monotonically to zero at infinity, the set

$$W(\mathcal{Z}, \psi) = \{ \omega \in \Omega : |\omega - Z_\alpha| < \psi([\alpha]) \text{ i.o. in } I \}$$

consists of those points in Ω that are *well approximable by $\mathcal{Z}$ relative to ψ* in the sup-norm.

In Dodson et al (1990) a lower bound for the Hausdorff dimension of such sets — and far more general ones — was obtained based on the assumption that the family $\mathcal{Z}$ is ubiquitous in Ω in the following sense. Let $\rho : (0,\infty) \to (0,\infty)$ be another function converging to zero at infinity. Then $\mathcal{Z}$ is *ρ-ubiquitous in Ω*, if

$$\left| \Omega - \bigcup_{[\alpha] \le N} B(Z_\alpha; \rho(N)) \right| \to 0 \quad \text{as} \quad N \to \infty,$$

where, for any subset Z and any $\rho > 0$,

$$B(Z; \rho) = \{ x \in \mathbb{R}^m : |x - Z| < \rho \}$$

denotes the neighborhood of radius ρ around Z with respect to the sup-norm.

In the following we write $\dim \mathcal{Z}$ and $\operatorname{codim} \mathcal{Z}$ for the common dimension and codimension of the subspaces in $\mathcal{Z}$.

Theorem (Dodson et al 1990). *Suppose $\mathcal{Z}$ is ρ-ubiquitous in Ω. Then*

$$\dim W(\mathcal{Z}, \psi) \ge \dim \mathcal{Z} + \gamma \operatorname{codim} \mathcal{Z},$$

where

$$\gamma = \min \left\{ 1, \limsup_{r \to \infty} \frac{\log \rho(r)}{\log \psi(r)} \right\}.$$

We apply this result to our set

$$W_0(\psi) = \{ \omega \in \Omega : |k \cdot \omega| < \psi(|k|) \text{ i.o. in } \mathbb{Z}^m \}$$

by considering the family $\mathcal{E} = \{ E_k : k \in I \}$ with $E_k = \{ x \in \mathbb{R}^m : k \cdot x = 0 \}$,

$I = \mathbb{Z}^m \setminus \{0\}$, and $[k] = |k|$. Since

$$|k \cdot \omega| < \delta \iff \|\omega - E_k\| < \delta / \|k\|,$$

$\|\cdot\|$ the euclidean norm, we have

$$W(\mathcal{E}, \tilde{\psi}) \subseteq W_0(\psi),$$

where $\tilde{\psi}$ is given by $\tilde{\psi}(r) = \psi(r)/mr$.

Let $\Omega = (-1,1)^m$ for simplicity — any other case is no more general. We claim that $\mathcal{E}$ is ρ-ubiquitous in Ω with

$$\rho(r) = \frac{m! \log r}{r^m}.$$

Postponing the proof for a moment we find that

$$\gamma = \limsup_{r\to\infty} \frac{\log \rho(r)}{\log \tilde{\psi}(r)} = \limsup_{r\to\infty} \frac{-m}{\dfrac{\log \psi(r)}{\log r} - 1} = \frac{m}{\lambda + 1},$$

where λ is the order of ψ at infinity. Hence,

$$\begin{aligned} \dim W_0(\psi) \geq \dim W(\mathcal{E}, \tilde{\psi}) &\geq \dim \mathcal{E} + \gamma \operatorname{codim} \mathcal{E} \\ &= m - 1 + \frac{m}{\lambda + 1} \end{aligned}$$

as stated. Actual equality will follow by a standard covering argument as in Section 3.

To prove ubiquity we recall a variant of Dirichlet's theorem on Diophantine approximations.

Lemma. *For every $\omega \in (-1,1)^m$ and every integer $N \geq 1$ there exists an integer vector $k \in \mathbb{Z}^m \setminus \{0\}$ such that*

$$|k \cdot \omega| \leq \frac{m!}{N^{m-1}}, \quad |k| \leq N.$$

Proof. Without loss of generality we may assume that ω has nonnegative components. The points $l \cdot \omega$ with $l \in \mathbb{N}_0^m$, $0 \leq |l| \leq N$ all lie in the interval $[0, N]$, and there are exactly

$$\operatorname{card}\left\{ l \in \mathbb{N}_0^m : |l| \leq N \right\} = \binom{N+m}{m} \geq \frac{N^m}{m!}$$

of them, as one easily shows by induction. Hence, by Dirichlet's pigeon hole principle, there must be two distinct l' and l'' in this collection with

$$0 \leq \left| l' \cdot \omega - l'' \cdot \omega \right| \leq N \cdot \frac{m!}{N^m} = \frac{m!}{N^{m-1}}.$$

Setting $k = l' - l''$ we have $0 \neq |k| \leq N$ and the required estimate. ∎

The lemma together with the equivalence above shows that for any given $N \geq 1$, every $\omega \in \Omega$ falls into at least one of the strips $B(E_k; \rho_k)$, $0 < |k| \leq N$, when

$$\rho_k = \frac{m!}{\|k\|\, N^{m-1}}.$$

For $\|k\| \geq N/\log N$, we have $\rho_k \leq m! \log N / N^m = \rho(N)$, and so those strips are covered by the bigger strips $B(E_k; \rho(N))$. Consequently,

$$\begin{aligned} \Big| \Omega - \bigcup_{[k] \leq N} B(E_k; \rho(N)) \Big| &\leq \Big| \bigcup_{\|k\| \leq N/\log N} B(E_k; \rho_k) \Big| \\ &\ll \sum_{\|k\| \leq N/\log N} \frac{1}{\|k\|\, N^{m-1}} \\ &\ll \sum_{r \leq N/\log N} \frac{r^{m-2}}{N^{m-1}} \\ &\ll \frac{1}{\log^{m-1} N}, \end{aligned}$$

where $\ll$ indicates inequality with a positive constant factor. The last term converges to zero as N tends to infinity, thus establishing ρ-ubiquity.

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