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On the Inclusion of Analytic Symplectic Maps in Analytic Hamiltonian Flows and Its Applications

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0 Introduction

Everybody knows that the time-1-shift of the flow of a hamiltonian vector field is a symplectic diffeomorphism homotopic to the identity. Moreover, if the underlying symplectic structure is exact, then this diffeomorphism is exact symplectic. Thus one may ask what the set of all maps arising this way looks like. That is, which exact symplectic diffeomorphisms homotopic to the identity can be included in the flow of a hamiltonian vector field?

Asked in such a global manner, the answer to this question is unknown, and we will not try to tackle it, either. Instead, we restrict ourselves to a local, perturbative situation. Given a small, exact symplectic perturbation of some integrable map, is this map included in the flow of some hamiltonian vector field? This vector field may be autonomous or non-autonomous. In the latter case, however, we ask the time dependence to be of period 1.

Posing the question this way, the answer is essentially "yes". Within the category of *smooth* maps and flows the argument is actually fairly straightforward, using the technique of generating functions. See the note [5] by Raphaël Douady. Within the category of *real analytic* maps and flows, however, the situation is less clear and more involved. Here, the first affirmative answer was given again by Raphaël Douady in his thesis [6] in a more qualitative manner. A more explicit, quantitative construction was recently given by the first author in [12].

Unfortunately, both these references are not easily available, and the argument in [12] is quite cumbersome. In this note we therefore want to describe one more version of this analytic interpolation theorem, which is more quantitative and explicit than the result in [5], yet whose proof is — we think — more straightforward and comprehensible than the one in [12].

There are at least two reasons for being interested in this interpolation problem. First, the problem is interesting in itself. But second, there is a significant number of results in the analytic perturbation theory of symplectic maps and hamiltonian flows, which are parallel and almost identical, yet are proven independently. But while it is easier to think a problem over in terms of maps, it is usually simpler to give a proof in terms of flows. Using interpolation, one can make use of this advantage and avoid redoing lengthy proofs for maps.

We illustrate this point by deriving an exponential stability estimate — also known as Nekhoroshev estimate — for almost integrable exact symplectic maps from the corresponding estimate for flows. The latter are well established, and we refer to the original paper by Nekhoroshev [18], the recent papers [13,19], and the references therein. For maps the corresponding result was conjectured in [18], and some special cases were recently proven in [3] and elsewhere. The proofs, however, are quite technical. By interpolation, they are greatly shortened.

We conclude this introduction by describing the two kinds of integrable maps whose perturbations we are going to study.

In the first case, let $D \subset \mathbb{R}^n$, $n \geq 1$, be a bounded, convex domain, let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the usual n-torus, and let $D \times \mathbb{T}^n$ be the symplectic manifold endowed with the standard exact symplectic structure $\upsilon = d\alpha$, where $\alpha = \sum_j I_j d\phi_j$. Suppose the map

$$F_0\colon\ D \times \mathbb{T}^n \to D \times \mathbb{T}^n, \qquad (I, \phi) \mapsto (I, \phi + \omega(I))$$

is real analytic and symplectic. Then $0 = \sum_j d\omega_j(I) \wedge dI_j = d\left(\sum_j \omega_j dI_j\right)$, so by convexity there exists a real analytic function h on D such that

$$\omega(I) = \frac{\partial h}{\partial I}.$$

The function h defines an integrable hamiltonian system on $D \times \mathbb{T}^n$ with equations of motions $\dot I = 0$, $\dot\phi = \partial_I h = \omega(I)$. Hence its flow X_h^t interpolates F_0. That is, we have

$$X_h^1 = X_h^t\big|_{t=1} = F_0$$

on $D \times \mathbb{T}^n$.

In the second case, let $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ be the euclidean $2n$-space with coordinates $z = (x, y)$ endowed with the standard exact symplectic structure

$$\upsilon = \sum\nolimits_j dx_j \wedge dy_j = \langle J\,\cdot\,, \,\cdot\,\rangle, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the usual scalar product and I the n-dimensional identity matrix. Suppose the linear map

$$F_*\colon\ \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \qquad z \mapsto \Lambda z$$

is symplectic. It is a standard fact, recalled in Appendix B, that one can write $\Lambda = \exp(JA)$ with a real symmetric matrix A, provided that either the spectrum of Λ contains no zero or negative eigenvalues, or Λ is the square of another symplectic matrix. The hamiltonian $h = \frac{1}{2}\langle Az, z\rangle$ with equations of motions $\dot z = JAz$ then gives rise to a flow X_h^t on $\mathbb{R}^{2n}$ such that

$$X_h^t\big|_{t=1} = \exp(JA) = F_*$$

as required.

In both these cases the interpolating hamiltonian is autonomous. This, however, is *not typical*, and in general one has to allow for *time-dependent* hamiltonians. Consider, for example, a real analytic, exact symplectic perturbation F of a one degree of freedom annulus map $F_0\colon (I, \phi) \mapsto (I, \phi + \omega(I))$. If the interpolating hamiltonian were autonomous, then it would be *integrable*, and so F would be integrable, too. But generically, arbitrarily small perturbations of F_0 give rise to transversal homoclinic intersections, thus destroying integrability [10,23]. So generically, the interpolating hamiltonian has to be time-dependent. — It is an open problem how to characterize all those maps which arise from *autonomous* hamiltonian flows.

1 Statement of the Interpolation Theorems

Again, let $D \subset \mathbb{R}^n$ be bounded and convex. We consider a real analytic *family* of real analytic maps

$$F_\epsilon\colon\ D \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n, \qquad -\epsilon_0 < \epsilon < \epsilon_0,$$

perturbing an integrable map $F_0\colon (I, \phi) \mapsto (I, \phi + \omega(I))$, where $\omega(I) = \partial h/\partial I$ with some real analytic function h on D. Moreover, the F_ϵ are assumed to be *exact* *symplectic*: there exists a family of differentiable functions W_ϵ on $D \times \mathbb{T}^n$ such that, for $\alpha = \sum_j I_j d\phi_j$,

$$F_\epsilon^* \alpha - \alpha = dW_\epsilon, \qquad -\epsilon_0 < \epsilon < \epsilon_0.$$

The domains of analyticity are assumed to be *uniform* and *independent* of ϵ. That is, each F_ϵ is assumed to be real analytic on a fixed complex neighbourhood $V_r D \times V_r \mathbb{T}^n$ of $D \times \mathbb{T}^n$, where

$$V_r D = \bigcup_{I_0 \in D} \left\{ I \in \mathbb{C}^n : |I - I_0| < r \right\} \subset \mathbb{C}^n,$$

$|\cdot|$ the euclidean norm, and likewise $V_r \mathbb{T}^n$. The same domain of analyticity is assumed for h. Possibly decreasing r and ϵ_0 we then also have a uniform bound for the sup-norms $|F_\epsilon|_{V_r D \times V_r \mathbb{T}^n}$ for all ϵ. Moreover,

$$|F_\epsilon - F_0|_{V_r D \times V_r \mathbb{T}^n} = O(\epsilon)$$

by Cauchy's estimate.

Theorem 1. *Suppose F_ϵ satisfies the preceding assumptions. Then for all sufficiently small ϵ there exists a real analytic, 1-periodic time dependent hamiltonian H_ϵ on $D \times \mathbb{T}^{n+1}$, such that*

$$X_{H_\epsilon}^t \big|_{t=1} = F_\epsilon$$

on $D \times \mathbb{T}^n$. Moreover, there exists a $\rho > 0$ such that H_ϵ is real analytic in $V_\rho D \times V_\rho \mathbb{T}^{n+1}$ for all small ϵ and satisfies

$$|H_\epsilon - h|_{V_\rho D \times V_\rho \mathbb{T}^{n+1}} = O(\epsilon).$$

as $\epsilon \to 0$.

In general, the flow $X_{H_\epsilon}^t$ of H_ϵ does *not* stay in $D \times \mathbb{T}^n$. But its time-1-map is still well defined for all small ϵ, since H_ϵ extends to a uniform neighbourhood of $D \times \mathbb{T}^n$ (that is, a δ-neighbourhood with respect to some norm).

For symplectic maps near fixed points the corresponding result is simpler to state. Let

$$F\colon\ U \to \mathbb{R}^{2n}, \quad z \mapsto \Lambda z + \hat{F}(z), \qquad \hat{F}(z) = O(z^2)$$

be a real analytic, symplectic map in some neighbourhood U of the origin in $\mathbb{R}^{2n}$.

Theorem 2. *Suppose that $\Lambda = \exp(JA)$ with some real symmetric matrix A. Then there exists a 1-periodic time-dependent hamiltonian*

$$H = \frac{1}{2}\langle Az, z\rangle + \hat{H}(z,t), \qquad \hat{H}(z,t) = O(z^3),$$

which is jointly real analytic in some neighbourhood of the origin in $\mathbb{R}^{2n}$ and in t, such that

$$X_H^t \big|_{t=1} = F$$

on that neighbourhood.

The proof of Theorem 2 is a variation of the proof of Theorem 1. In the following we therefore focus attention on the latter and add a few remarks on the former.

It would be natural to construct the interpolating time-dependent hamiltonian H_ϵ directly on the phase space $P = D \times \mathbb{T}^{n+1}$, for example by some kind of implicit function theorem. However, we are *not* able to do this. Instead, we first construct another family of real analytic hamiltonians $\bar{H}_\epsilon$ on the *extended phase space*

$$\bar{P} = \bar{D} \times \mathbb{T}^{n+1}, \quad \bar{D} = D \times \mathbb{R},$$

together with a family of symplectic section surfaces Σ_ϵ in the energy levels $\bar{H}_\epsilon = 0$ such that the induced Poincaré map Φ_ϵ of the hamiltonian flow is well defined and conjugate to F_ϵ, for all small ϵ. Then H_ϵ is found by the standard procedure of reducing $\bar{H}_\epsilon$ to its zero energy level [1,§45.B].

To set the stage let us first consider the unperturbed, integrable case. If the coordinates in $\bar{P}$ are denoted by I, E, ϕ, θ, then one naturally chooses the hamiltonian $\bar{H}_0 = h(I) + E$ with flow

$$X_{\bar{H}_0}^t\colon\ \bar{P} \to \bar{P}, \quad (I, E, \phi, \theta) \mapsto (I, E, \phi + t\omega(I), \theta + t).$$

Its Poincaré map Φ_0 with respect to the surface $\{\theta = 0\}$ is well defined and coincides with its time-1-shift: $\Phi_0 = X_{\bar{H}_0}^1$. Its further restriction to the symplectic, isoenergetic surface

$$\Sigma_0 = \left\{ \bar{H}_0 = 0,\ \theta = 0 \right\} \subset \bar{P}$$

is conjugate to F_0 in the sense that $\Phi_0 \circ j_0 = j_0 \circ F_0$, where j_0 is the canonical real analytic, symplectic embedding of P into $\bar{P}$, which is the identity in I and ϕ:

$$j_0\colon\ P \hookrightarrow \Sigma_0, \quad (I, \phi) \mapsto (I, -h(I), \phi, 0).$$

The isoenergetic reduction of $\bar{H}_0$ to $\{\bar{H}_0 = 0\}$ then yields $H_0 = h$.

To state the corresponding result for the family F_ϵ the following notation turns out to be convenient. We write

$$T_\epsilon\colon\ X \underset{\epsilon}{\longrightarrow} Y$$

for a family of real analytic maps T_ϵ from a bounded domain X in some euclidean space into another euclidean space containing Y, if first the T_ϵ have a — necessarily unique — analytic extension to a uniform neighbourhood of X for all small ϵ, and second, if for every $a > 0$ there exists a $b > 0$ such that

$$T_\epsilon\colon\ U_{a\epsilon}X \to U_{b\epsilon}Y \quad \text{and} \quad T_\epsilon\colon\ U_{-b\epsilon}X \to U_{-a\epsilon}Y$$

for all sufficiently small ϵ, depending on a. Here, $U_\rho X = V_\rho X \cap \mathbb{R}^m$ and $U_{-\rho}X = \mathbb{R}^m \backslash U_\rho(\mathbb{R}^m \backslash X)$ for $X \subset \mathbb{R}^m$.

For example,

$$F_\epsilon\colon P \underset{\epsilon}{\longrightarrow} P, \qquad F_\epsilon^{-1}\colon P \underset{\epsilon}{\longrightarrow} P,$$

since $F_\epsilon - F_0 = O(\epsilon)$. The composition of two families of such "ϵ-maps" is again a family of "ϵ-maps".

Theorem 3. *Suppose the assumptions of Theorem 1 hold. Then for all sufficiently small ϵ there exists a real analytic hamiltonian $\bar{H}_\epsilon$ on the extended phase space $\bar{P} = \bar{D} \times \mathbb{T}^{n+1}$ and a real analytic, symplectic embedding*

$$j_\epsilon\colon\ P \underset{\epsilon}{\longrightarrow} \Sigma_\epsilon = \left\{ \bar{H}_\epsilon = 0,\ \theta = 0 \right\} \subset \bar{P},$$

such that the Poincaré map Φ_ϵ of $\bar{H}_\epsilon$ is well defined for the embedded manifold Σ_ϵ and satisfies

$$\Phi_\epsilon \circ j_\epsilon = j_\epsilon \circ F_\epsilon .$$

Moreover, there exists a $\rho > 0$ such that $\bar{H}_\epsilon$ and j_ϵ extend analytically to $V_\rho \bar{P}$ and $V_\rho P$, respectively, for all sufficiently small ϵ, and satisfy

$$\left|\bar{H}_\epsilon - \bar{H}_0\right|_{V_\rho \bar{P}},\ \left|j_\epsilon - j_0\right|_{V_\rho P} = O(\epsilon)$$

as $\epsilon \to 0$.

Theorem 1 follows from Theorem 3 by isoenergetic reduction [1]. By the estimates we have

$$\frac{\partial \bar{H}_\epsilon}{\partial E} = 1 + O(\epsilon)$$

uniformly on $V_{\rho/2}\bar{P}$. Hence the equation $\bar{H}_\epsilon = 0$ can be solved for E everywhere by

$$E = -H_\epsilon(I, \phi, \theta), \qquad \text{or} \qquad H_\epsilon(I, \phi, \theta) + E = 0,$$

where H_ϵ is real analytic in some smaller fixed neighbourhood of $D \times \mathbb{T}^{n+1}$ for all small ϵ, with $|H_\epsilon - h| = O(\epsilon)$. Moreover, we may introduce the angle θ as time instead of t, since $d\theta/dt = \partial\bar{H}_\epsilon/\partial E = 1 + O(\epsilon)$. On the hypersurface $\bar{H}_\epsilon = 0$ one then finds

$$\frac{\partial I}{\partial \theta} = -\frac{\partial H_\epsilon}{\partial \phi}, \qquad \frac{\partial \phi}{\partial \theta} = \frac{\partial H_\epsilon}{\partial I}.$$

This proves Theorem 1.

Conversely, Theorem 3 obviously follows from Theorem 1, so the two are equivalent.

Theorem 3 is a slimmed down version of another, more quantitative interpolation theorem mentioned in the introduction. To state this result, assume that

$$|h|,\ |Dh|,\ \left|D^2 h\right| \le K$$

uniformly on $V_r D$ with respect to some fixed norms for vectors and matrices.

Theorem 4. *Suppose the map $F\colon P \to \mathbb{R}^n \times \mathbb{T}^n$ is real analytic in $V_r P$ and exact symplectic. If*

$$|F - F_0|_{V_r P} = \epsilon < \epsilon_0(K, n, r),$$

then there exists a real analytic hamiltonian $\bar{H}$ on $\bar{P}$ with a regular energy surface $\{\bar{H} = 0\}$ and a real analytic, symplectic embedding

$$j\colon\ P \hookrightarrow \Sigma = \left\{ \bar{H} = 0,\ \theta = 0 \right\},$$

such that the Poincaré map Φ of $\bar{H}$ is well defined on the embedded surface and satisfies $\Phi \circ j = j \circ F$. Moreover, $\bar{H}$ and j are real analytic on $V_\rho \bar{P}$ and $V_\rho P$, respectively, and satisfy

$$\left|\bar{H} - \bar{H}_0\right|_{V_\rho \bar{P}},\ \ |j - j_0|_{V_\rho P} \le c\epsilon,$$

where ρ and c only depend on K, n, r. In particular, $\rho = \dfrac{r}{(1+3K)^2}$.

Theorem 4 is proven in [12]. It sharpens the original result by Douady [6] in two important respects. First, the "smoothness of the interpolating hamiltonian is under control" — that is, there is an explicit control of the radius of analyticity ρ in terms of r and K. Second, $H - H_0$ and $j - j_0$ are shown to be of the order of $F - F_0$, and not just to converge to zero as $F - F_0$ converges to zero. These improvements are essential for more quantitative applications such as Nekhoroshev estimates.

Theorem 3 is simpler to prove than Theorem 4 as the map F is included in a real analytic family F_ϵ of maps. Thus we are able to state the result "for all sufficiently small ϵ" without making the smallness condition explicit. The proof itself is based on the Grauert embedding theorem and mostly follows the rather natural scheme proposed by Douady [6].

2 Applications

1. Nekhoroshev Estimates. As a first application of the preceding results we derive an exponential stability estimate for symplectic maps from corresponding estimates for hamiltonian flows, which are also known as Nekhoroshev estimates.

Again, let $D \subset \mathbb{R}^n$ be bounded and convex, and consider a family

$$F_\epsilon\colon\ D \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n, \qquad -\epsilon_0 < \epsilon < \epsilon_0,$$

of real analytic, exact symplectic maps as in Theorem 1. In particular, F_ϵ is a perturbation of an integrable map

$$F_0\colon\ D \times \mathbb{T}^n \to D \times \mathbb{T}^n, \quad (I, \phi) \mapsto (I, \phi + \omega(I)),$$

where $\omega(I) = \partial h/\partial I$ with some real analytic function h on D.

We call F_0 *m-steep on* D, if its extended hamiltonian

$$\bar{h} = h(I) + E$$

is steep on $\bar{D} = D \times \mathbb{R}$ as defined in [18] or [14]. We do not repeat this definition here, however, as it is quite technical and not very illuminating. Moreover, an elegant, sufficient characterization of steepness may be found in [11].

We just observe the following. The map F_0 is in particular m-steep on D, if h is strictly *convex* on D, because then $\bar{h}$ is quasi-convex on $\bar{D}$. Moreover, steepness is a generic property in the following sense. The set of all non-steep functions h is given by infinitely many polynomial relations in the Taylor coefficients of H at any given point in D. The set of non-steep functions thus has infinite codimension [18].

With every initial position (I_0, ϕ_0) in $D \times \mathbb{T}^n$ we associate its — possibly finite — orbit

$$(I_n, \phi_n) = F_\epsilon^n(I_0, \phi_0),$$

which is defined as long as its iterates (I_n, ϕ_n) are again in $D \times \mathbb{T}^n$ for positive or negative n. For simplicity, we do not indicate its dependence on ϵ. For $\epsilon = 0$, we have $I_n = I_0$ for all integers n on every orbit. For $\epsilon \neq 0$ the result is the following.

Theorem 5. *Suppose F_ϵ satisfies the assumptions of Theorem 1. If F_0 is m-steep on D, then for all sufficiently small ϵ one has*

$$|I_n - I_0| \le c_1\epsilon^b \quad \textit{for} \quad |n| \le c_2 \exp\left(c_3\epsilon^{-a}\right)$$

for every initial position (I_0, ϕ_0) in $D \times \mathbb{T}^n$, where the positive constants a and b depend on h and n, while the constants c_1, c_2, c_3 depend on h, n and the analyticity properties of F_ϵ. In particular, one has

$$a = \frac{1}{2n+2} = b,$$

if h is strictly convex.

Proof. Interpolating F_ϵ with the help of Theorem 3 we obtain a family $\bar{H}_\epsilon$ of real analytic perturbations of $\bar{H}_0 = h + E$. The latter is steep by assumption, and in particular quasi-convex, if h is strictly convex. So the estimates of [18,13,19] apply, giving the exponential stability of the flow of $\bar{H}_\epsilon$:

$$|I(t) - I(0)| \le c_1\epsilon^b \quad \text{for} \quad |t| \le c_2 \exp\left(c_3\epsilon^{-a}\right)$$

for every initial position in $\bar{D} \times \mathbb{T}^{n+1}$. This implies the corresponding estimate for F_ϵ with a different c_1, since the embedding j_ϵ is ϵ-close to j_0. ∎

Instead of considering a family F_ϵ we may also consider a single real analytic, exact symplectic map F, requiring that $|F - F_0|_{V_r P}$ is sufficiently small for some $r > 0$. In this case we have to appeal to Theorem 4 instead of Theorem 3 in the proof below.

Theorem 5*. *Suppose F_0 is m-steep on D, and F satisfies the conditions of Theorem 4. If $|F - F_0|_{V_r P} = \epsilon \le \epsilon_0(h, n, r)$, then one has*

$$|I_n - I_0| \le c_1\epsilon^b \quad \textit{for} \quad |n| \le c_2 \exp\left(c_3\epsilon^{-a}\right)$$

for every initial position (I_0, ϕ_0) *in* $D \times \mathbb{T}^n$, *where the positive constants* a *and* b *depend on* h *and* n, *while* c_1, c_2, c_3 *depend on* h, n, r. *In particular,* $a = 1/(2n+2) = b$, *if* h *is strictly convex.*

This result was conjectured by Nekhoroshev, and stated as a theorem in [18], but no proof was given yet.

2. KAM-Theorems. For the same family F_ϵ of symplectic maps we may also derive, by interpolation, the well known classical KAM theorem about the persistence of invariant tori from the corresponding result for flows. In the analytic category this was first done by Douady [5] – indeed, this was the very motivation for his interpolation theorem. To this end we assume that the unperturbed integrable map F_0 is *nondegenerate on* D in the sense that

$$\det \frac{\partial^2 h}{\partial I^2} = \det \frac{\partial \omega}{\partial I} \neq 0$$

on D. This condition is also known as the twist condition.

Theorem 6. *Suppose* F_ϵ *satisfies the assumptions of Theorem 1. If* F_0 *is nondegenerate on* D, *then for all sufficiently small* ϵ *the map* F_ϵ *possesses a Cantor family of real analytic invariant* n*-tori on which the mapping is conjugate to a rigid translation* $\varphi \mapsto \varphi + \omega$. *Moreover, the measure of its complement in* $D \times \mathbb{T}^n$ *is of the order of* $\sqrt{\epsilon}$, *provided the boundary of* D *is piecewise smooth.*

Sketch of Proof. Again we interpolate F_ϵ by Theorem 3 with a family $\bar{H}_\epsilon$ of real analytic hamiltonians perturbing $\bar{H}_0 = h + E$. The latter is *isoenergetically nondegenerate* on $\bar{D}$, since

$$\det \begin{pmatrix} \partial^2 \bar{H}_0 & \partial \bar{H}_0 \\ \partial \bar{H}_0^T & 0 \end{pmatrix} = \det \begin{pmatrix} \partial^2 h & 0 & \partial h \\ 0 & 0 & 1 \\ \partial h^T & 1 & 0 \end{pmatrix} = -\det \partial^2 h \neq 0$$

on $\bar{D}$. Hence the isoenergetic KAM theorem [1,2] applies, and for sufficiently small ϵ the level set $\{\bar{H}_\epsilon = 0\}$ is filled, up to a set of measure $O(\sqrt{\epsilon})$, by an n-dimensional Cantor family of real analytic invariant $n+1$-tori, which is a $O(\sqrt{\epsilon})$-deformation of some trivial Cantor subfamily of invariant $n+1$-tori for $\bar{H}_0$. Moreover, on this family of tori there are real analytic coordinates $\bar{\varphi}$, depending smoothly on the torus, so that the flow is given by $\bar{\varphi} \mapsto \bar{\varphi} + \bar{\omega} t$ for some $\bar{\omega}$ parametrizing the tori.

In these coordinates, $\bar{\varphi}_{n+1} = 0$ defines a transversal family of n-tori, which fills a $2n$-dimensional section up to measure $O(\sqrt{\epsilon})$, and on which the induced flow map is a rigid translation. In the coordinates of the phase space of $\bar{H}_\epsilon$, this in turn defines a section S_ϵ which is $O(\sqrt{\epsilon})$-close to the section $\Sigma_\epsilon = \{\bar{H}_\epsilon = 0, \theta = 0\}$ considered in Theorem 3. Arguing as in section 4.3 below the map F_ϵ is also conjugated to the flow map on S_ϵ as well, and the conclusion of Theorem 6 follows. ∎

Along the same lines we may also obtain another type of KAM theorem for maps concerned with the preservation of *lower dimensional* tori. For hamiltonian flows such a theory was established recently – see [9,20] and the references therein – but for symplectic maps it seems to be new. For a reversible analogue see [21]

We briefly indicate the result. We consider a real analytic family F_ϵ of exactly symplectic maps perturbing the integrable map

$$\begin{aligned} F_0\colon\ D \times \mathbb{T}^n \times \mathbb{R}^{2m} &\to D \times \mathbb{T}^n \times \mathbb{R}^{2m} \\ (I, \phi, z) &\mapsto \big(I, \phi + \omega(I), \Lambda(I) z\big), \end{aligned}$$

where $n, m \geq 1$, $\omega(I) = \partial h / \partial I$ with some real analytic function h, and $\Lambda(I)$ is diagonal with eigenvalues

$$\lambda_1(I), \ldots, \lambda_m(I), \lambda_1(I)^{-1}, \ldots, \lambda_m(I)^{-1}.$$

This map admits an n-parameter family $\mathbb{T}^n \times \{I\} \times \{0\}$ of invariant n-tori filling completely a $2n$-dimensional submanifold of the $2n+2m$-dimensional phase space. The objective is to prove the persistence of a large subfamily of it for small $\epsilon \neq 0$.

To this end the following *nondegeneracy conditions* are imposed. First, the determinant $\det \partial^2 h / \partial I^2$ does not vanish identically on D. Second, also none of the functions

$$e^{ik\cdot\omega} - \lambda_j, \quad e^{ik\cdot\omega} - \frac{\lambda_j}{\lambda_l}, \quad e^{ik\cdot\omega} - \lambda_j \lambda_l$$

vanishes identically on D for all $k \in \mathbb{Z}^n$ and $1 \leq j, l \leq m$. – So in particular, no eigenvalue is identically equal to 1 or -1, and no two eigenvalues are identically equal. Note that the second condition is automatically satisfied for all large k, so it is in fact a *finite* condition.

Theorem 7. *Suppose* F_0 *satisfies the preceding assumptions, and* F_ϵ *is jointly real analytic in a uniform complex neighbourhood of* $\mathbb{T}^n \times D \times \{0\}$ *and* $-\epsilon_0 < \epsilon < \epsilon_0$. *Then for all sufficiently small* ϵ *there exists an* n*-parameter Cantor family of real analytic invariant* n*-tori filling some* $2n$*-dimensional submanifold in phase space up to a subset whose measure tends to zero with* ϵ. *Moreover, on each torus the map is* *conjugate to a rigid translation, and around it the variational equations (with discrete time) can be put into constant coefficient form.*

The proof consists in interpolating the map F_ϵ by combining the ideas of Theorems 1 and 2 and applying a slightly modified version of the KAM theorem on lower dimensional tori in [20]. We forego the technical details. We just point out that the measure of the complementary set can not be made more explicit because of the non-quantitative nature of the nondegeneracy conditions.

3. Another Application. As another possible application of the interpolation theorems we mention the problem of establishing the exponentially small splitting of separatrices in nearly integrable maps. For the standard map, for instance, one rather laborious approach was proposed by Lazutkin in 1984 – see [15]. However, the recent work of Gelfreich in St. Petersburg and Delshams and Seara in Barcelona made it clear that this approach is much simpler in the framework of vector fields. It is conceivable that by interpolation the splitting problem for the standard map is more easily solved to hamiltonian flows, although more work is still required.

3 Proof of Theorem 3

The proof proceeds in four steps. First, an abstract real analytic manifold is constructed, on which the interpolation problem is trivially solved. Second, this manifold is embedded into a manifold with action angle coordinates, but with a nonstandard symplectic structure. In the third step, this structure is put back into constant standard form by the Moser-Weinstein Theorem. Finally, the surface of section defining the Poincaré map is straightened out.

Step 1. Abstract Suspension Manifold. We introduce the space

$$A = D \times \mathbb{T}^n \times \mathbb{R} \times \Theta,$$

where $\Theta = (\theta_0, \theta_1)$ is some interval of length $\theta_1 - \theta_0 < 2$ containing $[0,1]$. The A-coordinates are denoted by $\tilde I, \tilde\phi, \tilde E, \tilde\theta$ to distinguish them from the usual action angle coordinates later on. We identify points in A through the map

$$T_\epsilon\colon\ (\tilde I, \tilde\phi, \tilde E, \tilde\theta) \mapsto (F_\epsilon(\tilde I, \tilde\phi), \tilde E, \tilde\theta - 1).$$

In particular, $T_0\colon (\tilde I, \tilde\phi, \tilde E, \tilde\theta) \mapsto (\tilde I, \tilde\phi + \omega(\tilde I), \tilde E, \tilde\theta - 1)$. We then have

$$T_\epsilon\colon\ A_+ \underset{\epsilon}{\longrightarrow} A_- \quad \text{and} \quad T_\epsilon^{-1}\colon\ A_- \underset{\epsilon}{\longrightarrow} A_+$$

for the subdomains $A_- = \{\tilde\theta_0 < \tilde\theta < \theta_1 - 1\}$ and $A_+ = \{\tilde\theta_0 + 1 < \tilde\theta < \tilde\theta_1\}$ of A by the analogous properties of the map F_ϵ. Note that A_- and A_+ are *disjoint*, since $\theta_1 - \theta_0 < 2$.

The map T_ϵ is real analytic and preserves the symplectic structure

$$\tilde\upsilon = \sum\nolimits_j d\tilde I_j \wedge d\tilde\phi_j + d\tilde E \wedge d\tilde\theta,$$

since F_ϵ is symplectic. Hence, the quotient space

$$B_\epsilon = A/T_\epsilon$$

is a real analytic manifold with a real analytic symplectic structure $\tilde\upsilon_\epsilon$. The hamiltonian $\tilde H = \tilde E$ is real analytic and well defined on B_ϵ, since it is T_ϵ-invariant, too. Its flow $X^t_{\tilde H}$ is simply a shift of the $\tilde\theta$-variable, and starting at $\tilde\theta = 0$ we find that

$$X^1_{\tilde H}(\tilde I, \tilde\phi, \tilde E, 0) = (\tilde I, \tilde\phi, \tilde E, 1) \cong (F_\epsilon(\tilde I, \tilde\phi), \tilde E, 0).$$

Thus, $X^1_{\tilde H} \circ \tilde\imath = \tilde\imath \circ F_\epsilon$ on P, where $\tilde\imath$ denotes the canonical symplectic embedding of P into the isoenergetic surface of section

$$\tilde S = \left\{ \tilde H = 0,\ \tilde\theta = 0 \right\} \subset \bar P.$$

That is, $\tilde\imath$ is the identity on corresponding I- and ϕ-coordinates. Obviously, $\tilde H$ and $\tilde\imath$ extend to uniform complex neighbourhoods of B_ϵ and P, respectively, for all sufficiently small ϵ. — This is the “abstract solution” of the interpolation problem.

Step 2. Embedding of B_ϵ. We now seek a real analytic embedding of B_ϵ into the symplectic space $\bar P = \bar D \times \mathbb{T}^{n+1}$, where $\bar D = D \times \mathbb{R}$. The $\bar P$-coordinates are the usual action angle coordinates I, E, ϕ, θ.

For $\epsilon = 0$ this embedding is given as the inverse of the map

$$\Gamma_0\colon\ \begin{cases} \tilde I = I \\ \tilde E = h(I) + E \\ \tilde\phi = \phi - \theta\omega(I) \\ \tilde\theta = \theta \end{cases}.$$

One checks that $T_0 \circ \Gamma_0 = \Gamma_0 \circ \sigma_\theta$, where $\sigma_\theta\colon \theta \mapsto \theta - 1$. So equivalent points are mapped into equivalent points, whence $\Gamma_0\colon\ \bar P \to B_0$. Obviously,

$$\bar{H}_0 = \tilde{H} \circ \varGamma_0 = h(I) + E,$$
$$S_0 = \varGamma_0^{-1}(\tilde{S}) = \{ h(I) + E = 0,\ \theta = 0 \},$$

and one easily calculates that

$$\upsilon_0 = \varGamma_0^* \tilde{\upsilon}_0 = \sum_j dI_j \wedge d\phi_j + dE \wedge d\theta,$$

as it ought to be. — For $\epsilon \neq 0$ the key is the following result.

Proposition 1. *For all sufficiently small ϵ there exists a map*

$$G_\epsilon\colon\ B_\epsilon \underset{\epsilon}{\longrightarrow} B_0,$$

which is jointly real analytic in the A-coordinates and ϵ in a fixed complex neighbourhood of $A \times \{0\}$ and reduces to the identity for $\epsilon = 0$. That is to say, there is a lift

$$\tilde{G}_\epsilon\colon\ A \underset{\epsilon}{\longrightarrow} A,$$

satisfying $\tilde{G}_\epsilon \circ T_\epsilon = T_0 \circ \tilde{G}_\epsilon$, having these analyticity properties and reducing to the identity for $\epsilon = 0$. Moreover, G_ϵ is the identity on the E-coordinate, and also $G_\epsilon^{-1}\colon B_0 \underset{\epsilon}{\longrightarrow} B_\epsilon$.

We then define

$$\varGamma_\epsilon = G_\epsilon^{-1} \circ \varGamma_0\colon\ \bar{P} \underset{\epsilon}{\longrightarrow} B_\epsilon.$$

We have $|\varGamma_\epsilon - \varGamma_0| = O(\epsilon)$, and similarly for their derivatives, which is again an abbreviation of the analogous statements for the lifted maps. Hence, for the transformed symplectic structure we find

$$\upsilon_\epsilon = \varGamma_\epsilon^* \tilde{\upsilon}_\epsilon = \varGamma_0^* \tilde{\upsilon}_0 + O(\epsilon) = \upsilon_0 + O(\epsilon)\,.$$

All these estimates hold on some fixed complex neighbourhood of the real domains considered, uniformly for all small ϵ.

Proof of Proposition 1. In the following construction we may ignore the coordinate E, since it does not interfere with any other coordinate through T_ϵ and may thus be mapped identically.

Consider the real analytic manifold $\mathcal{A} = A \times \Delta$ with coordinates (X, δ), where $\Delta = (-\delta_0, \delta_0)$ denotes some small interval, and identify points via the map

$$\mathcal{T}\colon\ \begin{pmatrix} X \\ \delta \end{pmatrix} \mapsto \begin{pmatrix} T_\delta(X) \\ \delta \end{pmatrix}.$$

For Δ sufficiently small, $\mathcal{T}$ and $\mathcal{T}^{-1}$ are well defined and real analytic on the domains $A_+ \times \Delta$ and $A_- \times \Delta$, respectively, with analytic extensions to some uniform neighbourhoods of them. The quotient manifold $\mathcal{B} = \mathcal{A}/\mathcal{T}$ is thus real analytic, too, with a real analytic extension to $\mathcal{B}^o = \mathcal{A}^o/\mathcal{T}$, where $\mathcal{A}^o$ denotes some uniform neighbourhood of $\mathcal{A}$. Hence, by the Grauert embedding theorem [17], there exists a real analytic embedding

$$\mathcal{G}\colon\ \mathcal{B}^o \hookrightarrow \mathbb{R}^N$$

of this extended manifold into euclidean N-space, where N is a sufficiently large integer.

Consider now the submanifolds $\mathcal{B}^o_\epsilon = \{ (X, \delta) \in \mathcal{B}^o : \delta = \epsilon \}$ and their real analytic embeddings

$$\mathcal{G}_\epsilon = \mathcal{G}|_{\mathcal{B}^o_\epsilon}\colon\ \mathcal{B}^o_\epsilon \hookrightarrow \mathcal{M}^o_\epsilon \subset \mathbb{R}^N.$$

Let $\mathcal{B}_\epsilon = \mathcal{B}^o_\epsilon \cap \mathcal{B} \simeq B_\epsilon$, and let $\mathcal{M}_\epsilon \subset \mathcal{M}^o_\epsilon$ be the image of $\mathcal{B}_\epsilon \subset \mathcal{B}^o_\epsilon$. Around $\mathcal{M}^o_0$ we fix a tubular neighbourhood $\mathcal{U}$ and a real analytic normal projection $\mathcal{P}$ of $\mathcal{U}$ onto $\mathcal{M}^o_0$. Letting $\mathcal{P}_\epsilon = \mathcal{P}|_{\mathcal{M}_\epsilon}$, we have

$$\mathcal{P}_\epsilon\colon\ \mathcal{M}_\epsilon \underset{\epsilon}{\longrightarrow} \mathcal{M}_0, \qquad \mathcal{P}_\epsilon^{-1}\colon\ \mathcal{M}_0 \underset{\epsilon}{\longrightarrow} \mathcal{M}_\epsilon$$

for all sufficiently small ϵ, since $\mathcal{M}_\epsilon$ depends analytically on ϵ and reduces to $\mathcal{M}_0$ for $\epsilon = 0$. Thus we can define a mapping

$$G_\epsilon = \mathcal{G}_0^{-1} \circ \mathcal{P}_\epsilon \circ \mathcal{G}_\epsilon$$

from $\mathcal{B}_\epsilon \simeq B_\epsilon$ into $\mathcal{B}^o_0$ which reduces to the identity for $\epsilon = 0$.

This map can be lifted to a map $\tilde{G}_\epsilon$ of some uniform neighbourhood of A into some neighbourhood of A as follows. The middle portion of A, the domain $\{\theta_1 - 1 < \tilde{\theta} < \theta_0 + 1\}$, is projected *one-to-one* onto B_ϵ, and G_ϵ is close to the identity. So there is a *uniquely* defined lift on this part for all small ϵ, which is ϵ-close to the identity. Extending this lift continuously in the usual manner, it remains uniformly ϵ-close to the identity. Therefore, for all sufficiently small ϵ, it extends to a map $\tilde{G}_\epsilon$ from some small uniform neighbourhood of A into the uniform neighbourhood defining $\mathcal{A}^o$. Moreover, $\tilde{G}_\epsilon\colon\ A \underset{\epsilon}{\longrightarrow} A$.

The relation $\tilde{G}_\epsilon \circ T_\epsilon = T_0 \circ \tilde{G}_\epsilon$ is obvious for a lifting, and the claim about G_ϵ^{-1} also follows immediately. ∎

Step 3. Standard Symplectic Structure. We now put the symplectic structure υ_ϵ on $\bar{P}$ into standard form, using the Moser-Weinstein theorem [16]. To this end we need to know that υ_ϵ is *exact*.

Proposition 2. *For all sufficiently small ϵ there exists a 1-form α_ϵ on $\bar{P}$ such that $d\alpha_\epsilon = \upsilon_\epsilon$. Moreover, α_ϵ can be chosen to be real analytic jointly in the $\bar{P}$-coordinates and ϵ.*

Proof. We first observe that υ_ϵ is *smoothly* exact, since F_ϵ is exact symplectic. That is, for all small ϵ there exists a 1-form β_ϵ with *smooth* coefficients, such that $\upsilon_\epsilon = d\beta_\epsilon$. The proof is fairly easy and given in Appendix A.

By the abstract de Rham theorem [22] the existence of a smooth solution β_ϵ implies the existence of a real analytic solution α_ϵ, since the real analytic de Rham complex on a real analytic manifold is acyclic [4]. However, this existence result is not constructive, and in particular does not provide the analytic dependence of α_ϵ on the parameter ϵ. Therefore, we are going to construct α_ϵ explicitly, making use of specific properties of the manifold $\bar{P}$.

Fix some $\rho > 0$ so that υ_ϵ extends analytically to $U_\rho \bar{P}$ for all small ϵ. Since D is assumed to be convex, there exists a real analytic retraction

$$\pi_t\colon\ U_\rho \bar{P} \to U_\rho \bar{P}, \qquad 0 \le t \le 1,$$

such that $\pi_1 = id$, while π_0 retracts $\bar{D}$ to a single point $(I_0, 0)$ and leaves $\mathbb{T}^{n+1}$ untouched. The two 2-forms $\pi_1^* \upsilon_\epsilon = \upsilon_\epsilon$ and $\pi_0^* \upsilon_\epsilon = \upsilon'_\epsilon$ are then analytically cohomological. Indeed, there is an exact formula for a 1-form α'_ϵ such that

$$\upsilon_\epsilon = \upsilon'_\epsilon + d\alpha'_\epsilon,$$

and α'_ϵ is jointly analytic in the $\bar{P}$-coordinates and ϵ in a uniform neighbourhood of $\bar{P}$ [7].

The form υ'_ϵ is real analytic and closed. Its coefficients do not depend on I or E, and it vanishes on the tangent vectors $\partial/\partial I_j$, $1 \le j \le n$, and $\partial/\partial E$. Therefore,

$$\upsilon'_\epsilon = \sum_{i<j} a_{ij}(\bar{\phi})\, d\bar{\phi}_i \wedge d\bar{\phi}_j,$$

where $\bar{\phi} = (\bar{\phi}_1, \ldots, \bar{\phi}_{n+1}) = (\phi, \theta)$. Such a form is cohomological to its averaged form

$$\upsilon''_\epsilon = \sum_{i<j} b_{ij}\, d\bar{\phi}_i \wedge d\bar{\phi}_j, \qquad b_{ij} = \frac{1}{\operatorname{vol} \mathbb{T}^{n+1}} \int_{\mathbb{T}^{n+1}} a_{ij}(\bar{\phi})\, d\bar{\phi}.$$

That is, there exists a 1-form α''_ϵ, such that

$$\upsilon'_\epsilon = \upsilon''_\epsilon + d\alpha''_\epsilon,$$

and α''_ϵ is jointly analytic in the $\bar{P}$-coordinates and ϵ [7].

We thus arrive at a 2-form υ''_ϵ on $\mathbb{T}^{n+1}$ with constant coefficients, which we know is *differentiably* exact: $\upsilon''_\epsilon = d\beta''_\epsilon$ with some differentiable 1-form β''_ϵ. This, however, implies that υ''_ϵ *vanishes*. Consider, for example, the coefficient b_{12}. Averaging over the 2-torus $T = \{\bar{\phi}_3 = \cdots = \bar{\phi}_{n+1} = 0\}$ we find

$$b_{12} = \frac{1}{\operatorname{vol} T} \int_T \upsilon''_\epsilon = \frac{1}{\operatorname{vol} T} \int_T d\beta''_\epsilon = 0.$$

Thus $\upsilon''_\epsilon = 0$, whence $\upsilon_\epsilon = \upsilon'_\epsilon + d\alpha'_\epsilon = d\alpha'_\epsilon + d\alpha''_\epsilon$. This proves the proposition. ∎

We may now put the symplectic structure into standard form.

Proposition 3. *For all sufficiently small ϵ there exists a diffeomorphism $\Delta_\epsilon\colon\ \bar{P} \underset{\epsilon}{\longrightarrow} \bar{P}$, such that $\Delta_\epsilon^* \upsilon_\epsilon = \upsilon_0$. Moreover, $\Delta_0 = id$, and Δ_ϵ is jointly real analytic in the $\bar{P}$-coordinates and ϵ in a fixed complex neighbourhood of $\bar{P}$.*

Proof. This follows from the Moser-Weinstein theorem. Just observe that Δ_ϵ is given as the time-1-map of the real analytic, time-dependent vector field X_ϵ on $\bar{P}$ obtained as the solution of

$$i_{X_\epsilon} \upsilon^t_\epsilon = \alpha_\epsilon - \alpha_0,$$

where $\upsilon^t_\epsilon = (1-t)\upsilon_0 + t\upsilon_\epsilon$. ∎

Composing with the map Γ_ϵ we obtain, for all sufficiently small ϵ, a diffeomorphism

$$\Lambda_\epsilon = \Gamma_\epsilon \circ \Delta_\epsilon\colon\ \bar{P} \underset{\epsilon}{\longrightarrow} B_\epsilon, \qquad \Lambda_0 = \Gamma_0,$$

which is symplectic in the sense that

$$\Lambda_\epsilon^* \tilde{\upsilon}_\epsilon = \upsilon_0$$

for all these ϵ. Moreover, Λ_ϵ is jointly real analytic in the $\bar{P}$-coordinates and ϵ in a fixed complex neighbourhood of $\bar{P}$.

In B_ϵ the Poincaré map of the real analytic hamiltonian $\tilde{H}$ with respect to the isoenergetic section $\tilde{S}$ is well defined and interpolates the map F_ϵ. Indeed, it is simply the time-1-map of the hamiltonian $\tilde{H}$. The same is now true for their pull backs to $\bar{P}$. The time-1-map of the real analytic hamiltonian $\bar{H}_\epsilon = \tilde{H} \circ \Lambda_\epsilon$ is well defined on the surface $S_\epsilon = \Lambda_\epsilon^{-1}(\tilde{S})$ and interpolates F_ϵ. That is, we have a real analytic, symplectic embedding

$$i_\epsilon = \Lambda_\epsilon^{-1} \circ \tilde{\imath}: \; P \underset{\epsilon}{\longrightarrow} S_\epsilon \subset \bar{P}$$

such that $X^1_{\bar{H}_\epsilon} \circ i_\epsilon = i_\epsilon \circ F_\epsilon$ on P for all sufficiently small ϵ.

Step 4. Standard Surface of Section. The section S_ϵ is ϵ-close to the standard isoenergetic section

$$\Sigma_\epsilon = \left\{ \bar{H}_\epsilon = 0, \; \theta = 0 \right\} \subset \bar{P},$$

since they coincide for $\epsilon = 0$ and depend analytically on ϵ. Moreover, the vector field $X_{\bar{H}_\epsilon}$ is transversal to the hypersurface $\{\theta = 0\}$ in $\bar{P}$, since we have $\dot{\theta} = 1$ for $\epsilon = 0$. Hence there exists a well defined Poincaré flow map

$$\Psi_\epsilon \colon S_\epsilon \underset{\epsilon}{\longrightarrow} \Sigma_\epsilon, \qquad \Psi_\epsilon^{-1} \colon \Sigma_\epsilon \underset{\epsilon}{\longrightarrow} S_\epsilon$$

for all sufficiently small ϵ. This map is real analytic and symplectic, and $\Psi_0 = id$.

Thus we also have a real analytic, symplectic embedding

$$j_\epsilon = \Psi_\epsilon \circ i_\epsilon : \; P \underset{\epsilon}{\longrightarrow} \Sigma_\epsilon,$$

such that $\Phi_\epsilon \circ j_\epsilon = j_\epsilon \circ F_\epsilon$ on P, where

$$\Phi_\epsilon = \Psi_\epsilon \circ X^1_{\bar{H}_\epsilon} \circ \Psi_\epsilon^{-1} : \; \Sigma_\epsilon \underset{\epsilon}{\longrightarrow} \Sigma_\epsilon$$

is the Poincaré map of the hamiltonian vector field $X_{\bar{H}_\epsilon}$ with respect to Σ_ϵ. Finally,

$$\bar{H}_\epsilon \big|_{\epsilon=0} = \bar{H}_0 = h(I) + E, \qquad j_\epsilon \big|_{\epsilon=0} = j_0 = \Gamma_0^{-1} \circ \tilde{\imath}.$$

Both $\bar{H}_\epsilon$ and j_ϵ have a uniform radius of analyticity for all sufficiently small ϵ. The estimates of the theorem therefore follow from Cauchy's estimate, and the proof is finished.

4 Proof of Theorem 2

Rescaling coordinates by the dilation $z \mapsto \epsilon z$ we obtain a real analytic family of real analytic symplectic maps

$$F_\epsilon \colon \; P \to \mathbb{R}^{2n}, \qquad -\epsilon_0 < \epsilon < \epsilon_0,$$

where $P = \{|z| < r\} \subset \mathbb{R}^{2n}$ with some radius r to be chosen later, and

$$F_\epsilon(z) = \epsilon^{-1} F(\epsilon z) = \Lambda z + \epsilon^{-1} \hat{F}(\epsilon z) = \Lambda z + O(e\,|z|^2).$$

In particular, F_0 is linearly symplectic, and F_ϵ is conjugate to F on an ϵr-neighbourhood of the origin for all small $\epsilon \neq 0$. This family is interpolated in essentially the same way as in the preceding section, with a few modifications, which we indicate now.

Step 1. Abstract Suspension Manifold. We set $A = P \times \mathbb{R} \times \Theta$ and identify points through the map

$$T_\epsilon \colon \; (\tilde{z}, \tilde{E}, \tilde{\theta}) \mapsto (F_\epsilon(\tilde{z}), \tilde{E}, \tilde{\theta} - 1).$$

In particular, $T_0 \colon (\tilde{z}, \tilde{E}, \tilde{\theta}) \mapsto (\Lambda \tilde{z}, \tilde{E}, \tilde{\theta} - 1)$. The symplectic form $\tilde{\upsilon}$ is changed accordingly. Everything else remains the same.

Step 2. Embedding of B_ϵ. We now seek a real analytic embedding of $B_\epsilon = A/T_\epsilon$ into the space $\bar{P} = P \times \mathbb{R} \times \mathbb{T}$. For $\epsilon = 0$ this embedding is given as the inverse of the map

$$\Gamma_0 \colon \; \bar{P} \to B_0, \qquad \begin{cases} \tilde{z} = e^{-\theta JA} z \\ \tilde{E} = \frac{1}{2} \langle Az, z \rangle + E \\ \tilde{\theta} = \theta \end{cases},$$

which takes the symplectic structure on B_0 into the standard structure

$$\upsilon_0 = \sum\nolimits_j dx_j \wedge dy_j + dE \wedge d\theta$$

on $\bar{P}$. Moreover, we obtain $\bar{H}_0 = \frac{1}{2} \langle Az, z \rangle + E$. — For $\epsilon \neq 0$ we make use of the simple topology of $\bar{P}$ to normalize the maps G_ϵ in a useful way.

Addendum to Proposition 1. *The map G_ϵ may be defined so that*

$$T_0 \tilde{G}_\epsilon = id,$$

where $T_0 \tilde{G}_\epsilon$ denotes the tangential map of the lifting $\tilde{G}_\epsilon$ restricted to the tangent space of A along the submanifold $\{\tilde{z} = 0\}$.

With this provision we find that $\Gamma_\epsilon = G_\epsilon^{-1} \circ \Gamma_0$ takes the symplectic structure on B_ϵ into $\upsilon_\epsilon = \Gamma_\epsilon^* \tilde{\upsilon}_\epsilon = \upsilon_0 + O(\epsilon\,|z|)$. Moreover, the hamiltonian $\tilde{H} = \tilde{E}$ transforms into

$$\hat{H}_\epsilon = \frac{1}{2}\langle Az, z\rangle + E + O(\epsilon\,|z|^3),$$

since the E-coordinate transforms identically.

Proof of the Addendum. Again, we ignore the coordinate E. Otherwise, the construction of G_ϵ is modified in the following way.

Let $\mathcal{L} = T_0\pi$ be the tangential of the projection

$$\pi\colon\ \mathcal{A} \to \mathcal{A}, \quad (X, \delta) \mapsto (X, 0)$$

restricted to the submanifold $\mathcal{Z} = \{\tilde{z} = 0\}$. We have $T\mathcal{T} \circ \mathcal{L} = \mathcal{L} \circ T\mathcal{T}$, so $\mathcal{L}$ is also defined on the tangent bundle along $\mathcal{Z}/\mathcal{T} \simeq \mathbb{T}$ in B. Using the Grauert embedding map $\mathcal{G}$, this defines a corresponding map, again denoted by $\mathcal{L}$ for simplicity, on the tangent space to $\mathcal{M}$ along the image $\mathcal{N}$ of the submanifold $\mathcal{Z}$. Moreover, its restriction $\mathcal{L}_\epsilon$ to the tangent space of the slices $\mathcal{M}_\epsilon$ is a linear isomorphism.

As embedded manifolds, $\mathcal{M}$ and each of the $\mathcal{M}_\epsilon$ carry a natural real analytic riemannian metric, so that locally exponential maps are well defined. In particular, we can define an exponential map $\mathcal{E}_\epsilon$ from some neighbourhood of the zero section of the tangent bundle along $\mathcal{N} \cap \mathcal{M}_\epsilon \simeq \mathbb{T}$ orthogonal to the tangent bundle $T\mathcal{N}$, to some neighbourhood $\mathcal{U}_\epsilon$ of $\mathcal{N} \cap \mathcal{M}_\epsilon$ in $\mathcal{M}_\epsilon$. Moreover, those neighbourhoods $\mathcal{U}_\epsilon$ can be chosen as a δ-neighbourhood uniformly for all small ϵ.

This way we obtain a real analytic family of maps

$$\mathcal{E}_0 \circ \mathcal{L}_\epsilon \circ \mathcal{E}_\epsilon^{-1}\colon\ \mathcal{U}_\epsilon \underset{\epsilon}{\longrightarrow} \mathcal{U}_0,$$

whose tangential at $\mathcal{N}$ is equal to $\mathcal{L}_\epsilon$. Pulling them back to the abstract manifolds B_ϵ and choosing the radius r in the definition of P sufficiently small, we obtain a real analytic family of maps $G_\epsilon\colon B_\epsilon \underset{\epsilon}{\longrightarrow} B_0$, whose tangential at the submanifold $\mathcal{Z}$ is equal to $\mathcal{L}$, hence is the identity. ∎

Step 3. Standard Symplectic Structure. The 2-form υ_ϵ is closed on $\bar{P} = P \times \mathbb{R} \times \mathbb{T}$ and of the form $\upsilon_0 + O(\epsilon\,|z|)$. By the convexity of $P \times \mathbb{R}$ and a retraction argument, υ_ϵ is also exact on $\bar{P}$, with a 1-form

$$\alpha_\epsilon = \alpha_0 + O(\epsilon\,|z|^2),$$

where α_0 is any constant 1-form with $d\alpha_0 = \upsilon_0$. It follows that in the proof of Proposition 3 one obtains $X_\epsilon = O(\epsilon\,|z|^2)$, and consequently

$$\Delta_\epsilon = id + O(\epsilon\,|z|^2)$$

as its time-1-map.

Step 4. Final Transformations. We now proceed as before, observing that the transformed hamiltonian $\bar{H}_\epsilon = \hat{H}_\epsilon \circ \Delta_\epsilon$ is exactly of the same form as $\hat{H}_\epsilon$. Its isoenergetic reduction to $\bar{H}_\epsilon = 0$ is thus given by a hamiltonian

$$H_\epsilon(z, \theta) = \frac{1}{2}\langle Az, z\rangle + O(\epsilon\,|z|^3).$$

The time-1-map of H_ϵ interpolates F_ϵ on P for all sufficiently small ϵ. Fixing such an ϵ and scaling coordinates back we find that the time-1-map of

$$H = \epsilon^2 H_\epsilon(z/\epsilon, \theta) = \frac{1}{2}\langle Az, z\rangle + O(|z|^3)$$

interpolates F in an ϵr-neighbourhood of the origin. This completes the proof of Theorem 2.

A Smooth Exactness of υ_ϵ

We are going to show that the 2-form υ_ϵ is smoothly exact on $\bar{P}$. This is equivalent to showing that upstairs the 2-form $\tilde{\upsilon}$ is smoothly exact in B_ϵ. To simplify the notation we drop the ˜ and make the Ansatz

$$\beta_\epsilon = \alpha_s + E d\theta + dV_\epsilon, \qquad \alpha_s = \sum\nolimits_j I_j d\phi_j.$$

Then $d\beta_\epsilon = \upsilon$. The point is to choose the function V_ϵ so that $T_\epsilon^* \beta_\epsilon = \beta_\epsilon$ to make β_ϵ a 1-form on B_ϵ.

By hypotheses, F_ϵ is exact, so $F_\epsilon^* \alpha_s = \alpha_s + dW_\epsilon$ with some real analytic function W_ϵ. Hence,

$$\begin{aligned} T_\epsilon^* \beta_\epsilon &= F_\epsilon^* \alpha_s + E d\theta + T_\epsilon^* dV_\epsilon \\ &= \alpha_s + dW_\epsilon + E d\theta + T_\epsilon^* dV_\epsilon \\ &= \beta_\epsilon - dV_\epsilon + dW_\epsilon + d(V_\epsilon \circ T_\epsilon). \end{aligned}$$

So it suffices to find V_ϵ such that

$$V_\epsilon = W_\epsilon + V_\epsilon \circ T_\epsilon.$$

It is easy to solve this equation in terms of *smooth* functions, since A_- and $T_\epsilon^{-1}(A_-)$ are disjoint. If V^o is any smooth function on A_-, then set

$$V_\epsilon = \begin{cases} V^o & \text{on } A_- \\ W_\epsilon + V^o \circ T_\epsilon & \text{on } T_\epsilon^{-1}(A_-) \end{cases},$$

and extend V_ϵ smoothly to the rest of A. ▮

B The Logarithm of an Operator

Lemma B.1. *Let Λ be a nonsingular linear operator on $\mathbb{R}^m$.* (a) *There exists a linear operator B on $\mathbb{C}^m$ such that $\Lambda = \exp(B)$.* (b) *If Λ has no negative eigenvalues, then B can be chosen to be real.* (c) *The same holds, if Λ is the square of another linear operator on $\mathbb{R}^m$.*

Thus, a real logarithm always exists for the square of a nonsingular real operator. The converse is obviously also true: the existence of a real logarithm implies the existence of a real square root.

Proof. (a) Since Λ is nonsingular, there exists a contour Γ in $\mathbb{C}$ enclosing all the eigenvalues of Λ, but not 0. On Γ we may fix a branch of the complex logarithm to define the complex operator

$$B = \log \Lambda \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint_\Gamma \frac{\log z}{zI - \Lambda}\, dz.$$

Then $\exp(B) = \Lambda$ by the functional calculus of operators [8].

(b) In the absence of negative eigenvalues we can choose Γ to be symmetric to the real axis, but disjoint from its negative part. On Γ we may then choose the principal branch of the complex logarithm, so that we have $\overline{\log z} = \log \bar{z}$. The reality of B then follows by straightforward computation.

(c) Let $\Lambda = M^2$. Write $M = M_r \oplus M_c$, where M_r has only real and M_c only non-real eigenvalues. By (b) we have $M_r^2 = \exp(B_r)$ and $M_c = \exp(B_c)$ with real operators B_r and B_c. They commute, since M_r and M_c commute. Therefore,

$$\Lambda = \exp(B_r) \oplus \exp(2B_c) = \exp(B_r \oplus 2B_c),$$

and $B = B_r \oplus 2B_c$ is real. ▮

Let $m = 2n$ and endow $\mathbb{R}^{2n}$ with the standard symplectic structure J.

Lemma B.2. *Let Λ be symplectic. If the assumptions of (b) or (c) above are satisfied, then the real logarithm B of Λ is hamiltonian in the sense that $J^{-1}B$ is symmetric.*

Thus, $\Lambda = \exp(JA)$ with the real symmetric operator $A = J^{-1}B$.

Proof. We have to show equivalently that $JBJ = B^t$. By symplecticity of Λ, we have $J\Lambda J = -\Lambda^{-t}$, and by the definition of $B = \log \Lambda$,

$$JBJ = \frac{-1}{2\pi i} \oint_\Gamma \frac{\log z}{zI + J\Lambda J}\, dz = \frac{-1}{2\pi i} \oint_\Gamma \frac{\log z}{zI - \Lambda^{-t}}\, dz = -\log \Lambda^{-t}$$

Now, under the assumption of (b), we can choose Γ not to intersect the negative real axis, and the complex logarithm to be real on the positive real axis. This implies that $\log z^{-1} = -\log z$ on Γ, and thus $\log \Lambda^{-1} = -\log \Lambda$. It follows that $JBJ = B^t$. – The proof under the assumption of (c) is straightforward. ▮

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