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On the Korteweg - de Vries Equation and KAM Theory

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In this note we give an overview of results concerning the Korteweg-de Vries equation

$$u_t = -u_{xxx} + 6uu_x$$

and small perturbations of it. All the technical details will be contained in our forthcoming book [27].

The KdV equation is an evolution equation in one space dimension which is named after the two Dutch mathematicians Korteweg and de Vries [29], but was apparently derived even earlier by Boussinesq [10, 57]. It was proposed as a model equation for long surface waves of water in a narrow and shallow channel. Their aim was to obtain as solutions solitary waves of the type discovered in nature by Russell [58] in 1834. Later it became clear that this equation also models waves in other homogeneous, weakly nonlinear and weakly dispersive media. Since the mid-sixties the KdV equation received a lot of attention in the aftermath of the computational experiments of Kruskal and Zabusky [32], which lead to the discovery of the interaction properties of the solitary wave solutions and in turn to the understanding of KdV as an infinite dimensional integrable Hamiltonian system.

Our purpose here is to study small Hamiltonian perturbations of the KdV equation with periodic boundary conditions. In the unperturbed system *all* solutions are periodic, quasi-periodic, or almost periodic in time. The aim is to show that large families of periodic and quasi-periodic solutions persist under such perturbations. This is true not only for the KdV equation itself, but in principle for all equations in the KdV hierarchy. As an example, the second KdV equation will also be considered.

1 The KdV Equation

Let us recall those features of the KdV equation that are essential for our purposes. It was observed by Gardner [21] and Faddeev & Zakharov [18] that the KdV equation can be written in the Hamiltonian form

$$\frac{\partial u}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial H}{\partial u}$$

with the Hamiltonian

$$H(u) = \int_{S^1} \left(\tfrac{1}{2}u_x^2 + u^3\right) \mathrm{d}x,$$

where $\partial H/\partial u$ denotes the L^2-gradient of H. Since we are interested in *spatially periodic* solutions, we take as the underlying phase space the Sobolev space

$$\mathcal{H}^N = H^N(S^1;\mathbb{R}), \qquad S^1 = \mathbb{R}/\mathbb{Z},$$

of real valued functions with period 1, where $N \geq 1$ is an integer, and endow it with the Poisson bracket proposed by Gardner,

$$\{F,G\} = \int_{S^1} \frac{\partial F}{\partial u(x)} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial G}{\partial u(x)} \, \mathrm{d}x.$$

Here, F and G are differentiable functions on $\mathcal{H}^N$ with L^2-gradients in $\mathcal{H}^1$. This makes $\mathcal{H}^N$ a Poisson manifold, on which the KdV equation may also be represented in the form $u_t = \{u,H\}$ familiar from classical mechanics.

The initial value problem for the KdV equation on the circle S^1 is well posed on every Sobolev space $\mathcal{H}^N$ with $N \geq 1$: for initial data $u^{\mathrm{o}} \in \mathcal{H}^N$ it has been shown by Temam for $N = 1, 2$ [61] and by Saut & Temam for any real $N \geq 2$ [59] that there exists a unique solution evolving in $\mathcal{H}^N$ and defined globally in time. For further results on the initial value problem see for instance [60, 41, 44] as well as the more recent results [6, 7, 28].

The KdV equation admits infinitely many conserved quantities, or *integrals*, and there are many ways to construct such integrals [50, 51]. Lax [40] obtained a complete set of integrals in a particularly elegant way by considering the spectrum of an associated Schrödinger operator. For

$$u \in \mathcal{H}^0 = L^2 = L^2(S^1, \mathbb{R})$$

consider the differential operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u$$

on the interval $[0, 2]$ of *twice* the length of the period of u with periodic boundary conditions. It is well known [42] that its spectrum, denoted $\mathrm{spec}(u)$, is pure point and consists of an unbounded sequence of *periodic eigenvalues*

$$\lambda_0(u) < \lambda_1(u) \leq \lambda_2(u) < \lambda_3(u) \leq \lambda_4(u) < \dots \; .$$

Equality or inequality may occur in every place with a Ô≤Õ-sign, and one speaks of the *gaps* $(\lambda_{2n-1}(u), \lambda_{2n}(u))$ of the *potential* u and its *gap lengths*

$$\gamma_n(u) = \lambda_{2n}(u) - \lambda_{2n-1}(u), \qquad n \geq 1.$$

If some gap length is zero, then one speaks of a *collapsed gap*.

For $u = u(t, \cdot)$ depending also on t define the corresponding operator

$$L(t) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(t, \cdot).$$

Lax observed that u is a solution of the KdV equation if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t} L = [B, L],$$

where $[B, L] = BL - LB$ denotes the commutator of L with the *anti-symmetric* operator

$$B = -4\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 3u\frac{\mathrm{d}}{\mathrm{d}x} + 3\frac{\mathrm{d}}{\mathrm{d}x}u.$$

It follows by an elementary calculation that the flow of

$$\frac{\mathrm{d}}{\mathrm{d}t} U = BU, \qquad U(0) = I,$$

defines a family of *unitary* operators $U(t)$ such that $U^*(t)L(t)U(t) = L(0)$. Consequently, the spectrum of $L(t)$ is independent of t, and so the periodic eigenvalues $\lambda_n = \lambda_n(u)$ are conserved quantities under the evolution of the KdV equation. In other words, the flow of the KdV equation defines an *isospectral deformation* on the space of all potentials in $\mathcal{H}^N$.

From an analytical point of view, however, the periodic eigenvalues are not satisfactory as integrals, as λ_n is not a smooth function of u whenever the corresponding gap collapses. But McKean & Trubowitz [45] showed that the *squared gap lengths*

$$\gamma_n^2(u), \qquad n \geq 1,$$

together with the average

$$[u] = \int_{S^1} u(x)\,\mathrm{d}x$$

form another set of integrals, which are *real analytic* on all of L^2 and Poisson commute with each other. Moreover, the squared gap lengths together with the average determine uniquely the periodic spectrum of a potential [22].

The space L^2 thus decomposes into the isospectral sets

$$\mathrm{Iso}(u) = \left\{ v \in L^2 \colon \ \mathrm{spec}(v) = \mathrm{spec}(u) \right\},$$

which are invariant under the KdV flow and may also be characterized as

$$\mathrm{Iso}(u) = \left\{ v \in L^2 \colon \ \text{gap lengths}(v) = \text{gap lengths}(u), \ \ [v] = [u] \right\}.$$

As shown by McKean & Trubowitz [45] these are compact connected *tori*, whose dimension equals the number of positive gap lengths and is infinite generically.

Moreover, as the asymptotic behavior of the gap lengths characterizes the regularity of a potential in exactly the same way as its Fourier coefficients do [43], we have

$$u \in \mathcal{H}^N \quad \Leftrightarrow \quad \operatorname{Iso}(u) \subset \mathcal{H}^N$$

for each $N \geq 1$. Hence also the phase space $\mathcal{H}^N$ decomposes into a collection of tori of varying dimension which are *invariant* under the KdV flow.

2 Action-Angle and Birkhoff Coordinates

In classical mechanics the existence of a foliation of the phase space into Lagrangian invariant tori is tantamount, at least locally, to the existence of action-angle coordinates. This is the content of the Liouville-Arnold-Jost theorem. In the infinite dimensional setting of the KdV equation, however, the existence of such coordinates is far less clear as the dimension of the foliation is *nowhere* locally constant. Invariant tori of infinite and finite dimension each form dense subsets of the foliation. Nevertheless, action-angle coordinates *can* be introduced *globally* as we describe now. They will form the basis of our study of perturbations of the KdV equation.

To formulate the statement we define the phase spaces more precisely. For any integer $N \geq 0$, let

$$\mathcal{H}^N = \left\{ u \in L^2(S^1, \mathbb{R}) \colon \; \|u\|_N < \infty \right\},$$

where

$$\|u\|_N^2 = |\hat{u}(0)|^2 + \sum_{k \in \mathbb{Z}} |k|^{2N} \, |\hat{u}(k)|^2$$

is defined in terms of the discrete Fourier transform $\hat{u}$ of u. The Poisson structure $\{\cdot,\cdot\}$ is degenerate on $\mathcal{H}^N$ and admits the average $[\cdot]$ as a Casimir functional. The leaves of the corresponding symplectic foliation are given by $[u] = \text{const}$. Instead of restricting the KdV Hamiltonian to each leaf, it is more convenient to fix one such leaf, namely

$$\mathcal{H}_0^N = \left\{ u \in \mathcal{H}^N \colon \; [u] = 0 \right\},$$

which is symplectomorphic to each other leaf by a simple translation, and consider the mean value as a parameter. On $\mathcal{H}_0^N$ the Poisson structure is nondegenerate and induces a symplectic structure. Writing $u = v + c$ with $[v] = 0$ and $c = [u]$, the Hamiltonian then takes the form

$$H(u) = H_c(v) + c^3$$

with

$$H_c(v) = \int_{S^1} \left(\tfrac{1}{2} v_x^2 + v^3 \right) \mathrm{d}x + 6c \int_{S^1} \tfrac{1}{2} v^2 \, \mathrm{d}x.$$

We consider H_c as a 1-parameter family of Hamiltonians on $\mathcal{H}_0^N$.

We remark that

$$H^0 = \frac{1}{2} \int_{S^1} v^2 \, \mathrm{d}x$$

corresponds to translation and is the zero-th Hamiltonian of the KdV hierarchy.

To describe the action-angle variables on $\mathcal{H}_0^N$ we introduce the model space

$$\hbar_r = \ell_r^2 \times \ell_r^2$$

with elements (x, y), where

$$\ell_r^2 = \left\{ x \in \ell^2(\mathbb{N}, \mathbb{R}) \colon \; \|x\|_r^2 = \sum_{n \geq 1} n^{2r} \, |x_n|^2 < \infty \right\}.$$

We endow $\hbar_r$ with the standard Poisson structure, for which $\{x_n, y_m\} = \delta_{nm}$, while all other brackets vanish.

The following theorem was first proven in [1] and [2]. A quite different approach for this result – and the one we expand on here – was first presented in [26].

Theorem 2.1 *There exists a diffeomorphism $\Psi \colon \hbar_{1/2} \to \mathcal{H}_0^0$ with the following properties.*

(i) *Ψ is one-to-one, onto, bi-analytic, and preserves the Poisson bracket.*
(ii) *For each $N \geq 0$, the restriction of Ψ to $\hbar_{N+1/2}$, denoted by the same symbol, is a map $\Psi \colon \hbar_{N+1/2} \to \mathcal{H}_0^N$, which is one-to-one, onto, and bi-analytic as well.*
(iii) *The coordinates (x, y) in $\hbar_{N+1/2}$ are global Birkhoff coordinates for KdV. That is, for any $c \in \mathbb{R}$ the transformed Hamiltonian $H_c \circ \Psi$ depends only on $x_n^2 + y_n^2$, $n \geq 1$, with (x, y) being canonical coordinates.*

In the coordinate system (x, y) the KdV Hamiltonian H_c is a real analytic function of the actions I alone, where $I = (I_n)_{n \geq 1}$ with

$$I_n = \frac{1}{2}(x_n^2 + y_n^2).$$

This allows us to define its frequencies ω_c in the usual fashion:

$$\omega_c = (\omega_{c,n})_{n \geq 1}, \qquad \omega_{c,n} = \frac{\partial H_c}{\partial I_n},$$

and to establish them as real analytic functions of I.

These results are not restricted to the KdV Hamiltonian. They simultaneously apply to *every* real analytic Hamiltonian in the Poisson algebra of all Hamiltonians which Poisson commute with all action variables $I_1, I_2, \ldots$. In particular, one obtains action-angle coordinates for every equation in the KdV hierarchy. As an example, we will also consider the second KdV Hamiltonian later.

The existence of action-angle coordinates makes it evident that every solution of the KdV equation is *almost periodic* in time. In the coordinates of the model space every solution is given by

$$I(t) = I^{\mathrm{o}}, \qquad \theta(t) = \theta^{\mathrm{o}} + \omega(I^{\mathrm{o}})t,$$

where $(\theta^{\mathrm{o}}, I^{\mathrm{o}})$ corresponds to the initial data u^{o}, and $\omega(I^{\mathrm{o}}) = \omega_c(I^{\mathrm{o}})$ are the frequencies associated with I^{o} defined above. Hence, in the model space every solution winds around some underlying invariant torus

$$T_I = \left\{(x,y)\colon x_n^2 + y_n^2 = 2I_n, \ \ n \geq 1\right\},$$

determined by the initial actions I^{o}. Putting aside for the moment questions of convergence, the solution in the space $\mathcal{H}_0^N$ is thus winding around the embedded torus $\Psi(T_I)$ and is of the form

$$\begin{aligned} u(t) &= \Xi(\theta^{\mathrm{o}} + \omega(I^{\mathrm{o}})t, I^{\mathrm{o}}) \\ &= \sum_{k\in\mathbb{Z}^\infty, |k|<\infty} \Xi_k(I^{\mathrm{o}}) e^{2\pi \mathrm{i}\langle k,\theta^{\mathrm{o}}\rangle} e^{2\pi\mathrm{i}\langle k,\omega(I^{\mathrm{o}})\rangle t}. \end{aligned}$$

Here, $|k| = \sum_n |k_n|$, $\langle k,\theta\rangle = \sum_n k_n\theta_n$, and Ξ denotes the composition of the transformation from action-angle coordinates (θ, I) to Birkhoff coordinates (x,y) with Ψ, with each $\Xi_k(I^{\mathrm{o}})$ being an element of $\mathcal{H}_0^N$. Thus, every solution is almost-periodic in time.

We remark that the solution above can also be represented in terms of the Riemann theta function. The corresponding formula is due to Its & Matveev [13].

Among all almost-periodic solutions there is a dense subset of quasi-periodic solutions, which are characterized by a *finite* number of frequencies and correspond to finite gap potentials. To describe them more precisely, let $A \subset \mathbb{N}$ be a finite index set, and consider the set of *A-gap potentials*

$$\mathcal{G}_A = \left\{u \in \mathcal{H}_0^0\colon \gamma_n(u) > 0 \Leftrightarrow n \in A\right\}.$$

That is, $u \in \mathcal{G}_A$ if and only if precisely the gaps $(\lambda_{2n-1}(u), \lambda_{2n}(u))$ with $n \in A$ are open. Clearly,

$$u \in \mathcal{G}_A \quad \Leftrightarrow \quad \mathrm{Iso}(u) \subset \mathcal{G}_A,$$

and all finite gap potentials are smooth, in fact real analytic, as almost all gap lengths are zero. As might be expected there is a close connection between the set $\mathcal{G}_A$ and the subspace

$$\hbar_A = \left\{(x,y) \in \hbar_0\colon x_n^2 + y_n^2 > 0 \Leftrightarrow n \in A\right\}.$$

Addendum 1 (to Theorem 2.1). *The canonical transformation Ψ also has the following property.*

(iv) *For every finite index set $A \subset \mathbb{N}$, the restriction Ψ_A of Ψ to $\hbar_A$ is a map*

$$\Psi_A\colon\ \hbar_A \to \mathcal{G}_A,$$

which is one-to-one, onto, and bi-analytic.

It follows that $\mathcal{G}_A$ is a real analytic, invariant submanifold of $\mathcal{H}_0^0$, which is completely and analytically foliated into invariant tori $\mathrm{Iso}(u)$ of the same dimension $|A|$. The KdV flow consists of a quasi-periodic winding around each such torus which is characterized by $|A|$ frequencies ω_n, $n \in A$. The above representation reduces to a convergent, real analytic representation

$$u(t) = \sum_{k\in\mathbb{Z}^A} \Xi_k(I_A^{\mathrm{o}}) e^{2\pi\mathrm{i}\langle k,\theta_A^{\mathrm{o}}\rangle} e^{2\pi\mathrm{i}\langle k,\omega_A(I_A^{\mathrm{o}})\rangle t},$$

with $I_A = (I_n)_{n\in A}$, and similarly defined θ_A and ω_A.

3 Outline of the Proof of Theorem 2.1

The proof of the theorem splits into four parts. First we define actions I_n and angles θ_n for a potential q following a procedure well known for finite dimensional integrable systems. The formula for the actions I_n, due to Flaschka & McLaughlin [19], is given entirely in terms of the periodic spectrum of the potential. The angles θ_n, which linearize the KdV equation, were introduced even earlier by a number of authors, namely Dubrovin, Its, Krichever, Matveev, Novikov [12, 13, 15, 16, 25] (see also [14]), McKean & van Moerbeke [44], and McKean & Trubowitz [45, 46]. They are defined in terms of the Riemann surface $\Sigma(q)$ associated with the periodic spectrum of a potential. We show that each I_n is real analytic on L_0^2, while each θ_n, taken modulo 2π, is real analytic on the dense open domain $L_0^2 \setminus D_n$, where D_n denotes the subvariety of potentials with collapsed n-th gap.

Next, we define the cartesian coordinates x_n and y_n canonically associated to I_n and θ_n. Although defined originally only on $L_0^2 \setminus D_n$, we show that they extend real analytically to a complex neighbourhood W of L_0^2. Surely, the angle θ_n blows up when γ_n collapses, but this blow up is compensated by the rate at which I_n vanishes in the process. In particular, for real q the resulting limit will vanish.

Then we show that the thus defined map $\Omega\colon q \mapsto (x,y)$ is a diffeomorphism between L_0^2 and $\hbar_{1/2}$. The main problem here is to verify that $d_q\Omega$ is a linear isomorphism at *every* point q. This is done with the help of orthogonality relations among the coordinates, which are in fact their Poisson brackets. For the nonlinear Schrödinger equation the corresponding orthogonality relations have first been established by McKean & Vaninsky [47, 48]. It turned out that many of their ideas can also be used in the case of KdV.

We also verify that each Hamiltonian in the KdV hierarchy becomes a function of the actions alone, using their characterization in terms of the asymptotic expansion of the discriminant Δ as $\lambda \to -\infty$ and hence as spectral invariants.

Finally, we verify that Ω preserves the Poisson bracket. As it happens, it is more convenient to look at the associated symplectic structure. This way, we only need to establish the regularity of the gradient of θ_n at special points, not everywhere. Thus, we equivalently show that Ω is a symplectomorphism. This will complete the proof of the main results of Theorem 2.1.

4 Perturbations of the KdV Equation

Our aim is to investigate whether sufficiently small Hamiltonian perturbations of the KdV equation,

$$\frac{\partial u}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial H_c}{\partial u} + \varepsilon\frac{\partial K}{\partial u}\right),$$

admit almost-periodic solutions as well, which wind around invariant tori in phase space.

In the classical setting of integrable Hamiltonian systems of finitely many degrees of freedom this question is answered by the theory of Kolmogorov, Arnold and Moser, known as *KAM theory*. It states that with respect to Lebesgue measure the majority of the invariant tori of a real analytic, nondegenerate integrable system persist under sufficiently small, real analytic Hamiltonian perturbations. They are only slightly deformed and still completely filled with quasi-periodic motions. The base of this partial foliation of the phase space into invariant tori, however, is no longer open, but has the structure of a *Cantor set*: it is a nowhere dense, closed set with no isolated points.

Till now, there is no general infinite dimensional KAM theory to establish the persistence of infinite dimensional tori with almost periodic solutions for Hamiltonian systems arising from partial differential equations, such as the KdV equation. There does exist a KAM theory to this effect, but it is restricted to systems, in which the coupling between the different modes of oscillations is of short range type. See [53] and the references therein. Such a theory does not apply here, and we can not make any statement about the persistence of almost-periodic solutions.

But as noted above, there are also families of finite dimensional tori on real analytic submanifolds, corresponding to finite gap solutions and filling the space densely. In the classical setting a KAM theorem about the persistence of such lower dimensional tori was first formulated by Melnikov [49], and proven later by Eliasson [17]. It was independently extended to the infinite dimensional setting of partial differential equations by Kuksin [33, 35, 37]. In the following we describe this kind of perturbation theory for finite gap solutions, following [52, 54].

Again, let $A \subset \mathbb{N}$ be a *finite* index, and let $\varGamma \subset \mathbb{R}^A_+$ be a compact set of positive Lebesgue measure. We then set

$$\mathcal{T}_\varGamma = \bigcup_{I\in\varGamma} \mathcal{T}_I \subset \mathcal{G}_A, \qquad \mathcal{T}_I = \varPsi_A(T_I),$$

where

$$T_I = \big\{(x,y) \in \hbar_A \colon x_n^2 + y_n^2 = 2I_n,\ \ n \in A\big\} \cong \mathbb{T}^A \times \{I\},$$

with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the circle of length 2π. Notice that $\mathcal{T}_\Gamma \subset \bigcap_{N\geq 0} \mathcal{H}_0^N$ in view of the Addendum to Theorem 2.1.

We show that under sufficiently small real analytic Hamiltonian perturbations of the KdV equation the majority of these tori persists together with their translational flows, the tori being only slightly deformed. The following theorem was first proven by S. Kuksin for the case $c = 0$.

Theorem 4.2 *Let $A \subset \mathbb{N}$ be a finite index set, $\Gamma \subset \mathbb{R}_+^A$ a compact subset of positive Lebesgue measure, and $N \geq 1$. Assume that the Hamiltonian K is real analytic in a complex neighbourhood U of $\mathcal{T}_\Gamma$ in $\mathcal{H}_{0,\mathbb{C}}^N$ and satisfies the regularity condition*

$$\frac{\partial K}{\partial u} \colon\ U \to \mathcal{H}_{0,\mathbb{C}}^N, \qquad \left\|\frac{\partial K}{\partial u}\right\|_{N;U}^{\sup} = \sup_{u\in U}\left\|\frac{\partial K}{\partial u}\right\|_N \leq 1.$$

Then, for any real c, there exists an $\varepsilon_0 > 0$ depending only on A, N, c and the size of U such that for $|\varepsilon| < \varepsilon_0$ the following holds. There exist

(i) *a nonempty Cantor set $\Gamma_\varepsilon \subset \Gamma$ with* $\operatorname{meas}(\Gamma - \Gamma_\varepsilon) \to 0$ *as $\varepsilon \to 0$,*
(ii) *a Lipschitz family of real analytic torus embeddings*

$$\Xi \colon\ \mathbb{T}^n \times \Gamma_\varepsilon \to U \cap \mathcal{H}_0^N,$$

(iii) *a Lipschitz map $\chi \colon \Gamma_\varepsilon \to \mathbb{R}^n$,*

such that for each $(\theta, I) \in \mathbb{T}^n \times \Gamma_\varepsilon$, the curve $u(t) = \Xi(\theta + \chi(I)t, I)$ is a quasi-periodic solution of

$$\frac{\partial u}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial H_c}{\partial u} + \varepsilon\frac{\partial K}{\partial u}\right)$$

winding around the invariant torus $\Xi(\mathbb{T}^n \times \{I\})$. Moreover, each such torus is linearly stable.

Remark 4.3 Note that the L^2-gradient of a function on $\mathcal{H}_0^N$ has mean value zero by the definition of the gradient. On the other hand, the L^2-gradient of a function on the larger space $\mathcal{H}^N$ usually has man value different from zero, and the gradient of its restriction to $\mathcal{H}_0^N$ is the *projection* of the former onto $\mathcal{H}_0^N$:

$$\nabla\left(K|_{H_0^N}\right) = \operatorname{Proj}_{\mathcal{H}_0^N} \nabla K = \nabla K - [\nabla K],$$

with $\nabla = \partial/\partial u$. This, however, does not affect their Hamiltonian equations, since the derivative of the constant function $[\nabla K]$ vanishes. Therefore, we will not explicitly distinguish between these two gradients.

Remark 4.4 We already mentioned that $\mathcal{G}_A \subset \bigcap_{N\geq 0} \mathcal{H}_0^N$. Thus, the perturbed quasi-periodic solutions remain in $\mathcal{H}_0^N$ if the gradient of K is in $\mathcal{H}_0^N$.

Remark 4.5 The regularity assumption on K entails that K depends only on u, but not on its derivatives. So the perturbation effected by K is of *lower order* than the unperturbed KdV equation. In view of the derivation of the KdV equation as a model equation for surface waves of water in a certain regime expansion – see for example [63] – it would be interesting to obtain perturbation results which also include terms of higher order, at least in the region where the KdV approximation is valid. However, results of this type are still out of reach, if true at all.

Remark 4.6 We point out that the perturbing term $\varepsilon\partial K/\partial u$ need not be a differential operator. For example,

$$K(u) = \left(\int_{S^1} u^3 \,\mathrm{d}x\right)^2$$

has L^2-gradient

$$\frac{\partial K}{\partial u} = 6u^2 \int_{S^1} u^3 \,\mathrm{d}x,$$

to which the theorem applies as well.

Remark 4.7 The invariant embedded tori are linearly stable in the sense that the variational equations of motion along such a torus are reducible to constant coefficient form whose spectrum is located on the imaginary axis. Hence, all Lyapunov exponents of such a torus vanish.

Similar results hold for any equation in the KdV hierarchy. Consequently, for any given finite index set A, the manifold of A-gap potentials is foliated into the same family of invariant tori. The difference is only in the frequencies of the quasi-periodic motions on each of these tori. Therefore, similar results should also hold for the higher order KdV equations, once we can establish the corresponding nonresonance conditions.

As an example we consider the second KdV equation, which reads

$$\partial_t u = \partial_x^5 u - 10u\partial_x^3 u - 20\partial_x u \partial_x^2 u + 30u^2\partial_x u.$$

Its Hamiltonian is

$$H^2(u) = \int_{S^1} \left(\tfrac{1}{2}u_{xx}^2 + 5uu_x^2 + \tfrac{5}{2}u^4\right) \mathrm{d}x,$$

which is defined on $\mathcal{H}^2$. Again, with $u = v + c$, where $[v] = 0$, we get

$$H^2(u) = H_c^2(v) + \frac{5}{2}c^4$$

with

$$H_c^2(v) = H^2(v) + 10cH^1(v) + 30c^2H^0(v).$$

Here, $H^1(v) = \int_{S^1} \left(\frac{1}{2}v_x^2 + v^3\right) \mathrm{d}x$ is the familiar KdV Hamiltonian, and $H^0(v) = \frac{1}{2}\int_{S^1} v^2 \,\mathrm{d}x$ is the zero-th Hamiltonian of translation. We study this Hamiltonian on the space $\mathcal{H}_0^N$ with $N \geq 2$, considering c as a real parameter.

Theorem 4.8 *Let $A \subset \mathbb{N}$ be a finite index set, $\Gamma \subset \mathbb{R}_+^A$ a compact subset of positive Lebesgue measure, and $N \geq 3$. Assume that the Hamiltonian K is real analytic in a complex neighbourhood U of $\mathcal{T}_\Gamma$ in $\mathcal{H}_{0,\mathbb{C}}^N$ and satisfies the regularity condition*

$$\frac{\partial K}{\partial u}\colon\ U \to \mathcal{H}_{0,\mathbb{C}}^{N-2}, \qquad \left\|\frac{\partial K}{\partial u}\right\|_{N-2;U}^{\sup} \leq 1.$$

If $c \notin \mathcal{E}_A^2$, where the exceptional set $\mathcal{E}_A^2$ is an at most countable subset of the real line not containing 0 and with at most $|A|$ accumulation points, then the same conclusions as in Theorem 4.2 hold for the system with Hamiltonian $H_c^2 + \varepsilon K$.

Remark 4.9 The gradient $\partial K/\partial u$ is only required to be in $\mathcal{H}_0^{N-2}$. Still, the regularity assumption ensures that the perturbation is of *lower order* than the unperturbed equation.

We now give two simple examples of perturbations to which the preceding theorems apply. As a first example let

$$K(u) = \int_{S^1} F(x,u)\,\mathrm{d}x,$$

where F defines a real analytic map

$$\{\lambda \in \mathbb{R}\colon\ |\lambda| < R\} \to \mathcal{H}_0^N, \qquad \lambda \mapsto F(\,\cdot\,,\lambda),$$

for some $R > 0$ and $N \geq 1$. Then, with $f = \partial F/\partial\lambda$,

$$\frac{\partial K}{\partial u} = f(x,u) - [f(x,u)]$$

belongs to $\mathcal{H}_0^N$, and the perturbed KdV equation is

$$u_t = -u_{xxx} + 6uu_x + \varepsilon\frac{\mathrm{d}}{\mathrm{d}x}f(x,u).$$

Theorem 4.2 applies after fixing Γ and c for all sufficiently small ε. Of course, F may also depend on ε, if the dependence is, say, continuous.

We remark that perturbations of the KdV equation with K as above can be characterized equivalently as local perturbations given by $\frac{\mathrm{d}}{\mathrm{d}x}f(x,u(x))$, where f admits a power series expansion in the second argument,

$$f(x,\lambda) = \sum_{k\geq 0} f_k(x)\lambda^k,$$

convergent in $\mathcal{H}_0^N$. In this case, the perturbed equation is again a partial differential equation, and its Hamiltonian K is given by $K(u) = \int_{S^1} F(x,u)\,\mathrm{d}x$, where F is a primitive of f with respect to λ,

$$F(x,\lambda) = \sum_{k\geq 0} \frac{f_k(x)}{k+1}\lambda^{k+1}.$$

As a second example let

$$K(u) = \int_{S^1} F(x,u_x)\,\mathrm{d}x,$$

where F is as above with $N \geq 3$. More generally, F could also depend on u, but this adds nothing new. Then

$$\frac{\partial K}{\partial u} = -\frac{\mathrm{d}}{\mathrm{d}x} f(x,u_x)$$

belongs to $\mathcal{H}_0^{N-2}$, and the perturbed second KdV equation is

$$u_t = \cdots - \varepsilon \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x,u_x).$$

To this second example, Theorem 4.8 applies.

5 Outline of Proof of Theorems 4.2 and 4.8

A prerequisite for developing a perturbation theory of KAM type is the existence of coordinates with respect to which the variational equations along the unperturbed motions on the invariant tori reduce to constant coefficient form. Often, such coordinates are difficult to construct even locally. Here, they are provided *globally* by Theorem 2.1.

According to Theorem 2.1 the Hamiltonian of the KdV equation on the model space $\hbar_{3/2}$ is of the form

$$H_c = H_c(I_1, I_2, \ldots), \qquad I_n = \frac{1}{2}(x_n^2 + y_n^2).$$

The equations of motion are thus

$$\dot{x}_n = \omega_n(I)y_n, \qquad \dot{y}_n = -\omega_n(I)x_n,$$

with frequencies

$$\omega_n = \omega_{c,n} = \frac{\partial H_c}{\partial I_n}(I), \qquad I = (I_n)_{n\geq 1},$$

that are constant along each orbit. So each orbit is winding around some invariant torus $T_I = \big\{(x,y)\colon x_n^2 + y_n^2 = 2I_n,\ n \geq 1\big\}$, where the parameters $I = (I_n)_{n\geq 1}$ are the actions of its initial data.

We are interested in a perturbation theory for families of *finite*-dimensional tori T_I. So we fix an index set $A \subset \mathbb{N}$ of finite cardinality $|A|$, and consider tori with

$$I_n > 0 \quad \Leftrightarrow \quad n \in A.$$

The linearized equations of motion along any such torus have now constant coefficients and are determined by $|A|$ *internal frequencies* $\omega = (\omega_n)_{n\in A}$ and infinitely many *external frequencies* $\Omega = (\omega_n)_{n\notin A}$. Both depend on the $|A|$-dimensional parameter

$$\xi = (I_n^{\mathrm{o}})_{n\in A},$$

since all other components of I vanish in this family.

The KAM theorem for such families of finite dimensional tori requires a number of assumptions, among which the most notorious and unpleasant ones are the so called *nondegeneracy* and *nonresonance conditions*. In this case, they essentially amount to the following. First, the map

$$\xi \mapsto \omega(\xi)$$

from the parameters to the internal frequencies has to be a local homeomorphism, which is Lipschitz in both directions. In the classical theory this is known as *Kolmogorov's condition*. Second, the zero set of any of the frequency combinations

$$\langle k, \omega(\xi)\rangle + \langle l, \Omega(\xi)\rangle$$

has to be a set of measure zero, for each $k \in \mathbb{Z}^A$ and $l \in \mathbb{Z}^{\mathbb{N}\setminus A}$ with $1 \leq |l| \leq 2$. This is sometimes called *Melnikov's condition*.

The verification of these conditions for the KdV Hamiltonian requires some knowledge of its frequencies. One way to obtain this knowledge is to use Riemann surface theory: Krichever proved that the frequency map $\xi \mapsto \omega(\xi)$ is a local diffeomorphism everywhere on the space of A-gap potentials, see [3, 31], and Bobenko & Kuksin showed that the second condition is satisfied in the case $c = 0$ using Schottky uniformization [4]. Here, however, we follow a different and more elementary route to verify these conditions by computing the first coefficients of the Birkhoff normal form of the KdV Hamiltonian, which we explain now.

In classical mechanics the Birkhoff normal form allows to view a Hamiltonian system near an elliptic equilibrium as a small perturbation of an integrable system. This tool is also applicable in an infinite dimensional setting as ours. Writing

$$u = \sum_{n \neq 0} \gamma_n q_n e^{2\pi i n x}$$

with weights $\gamma_n = \sqrt{2\pi |n|}$ and complex coefficients $q_{\pm n} = (x_n \mp \mathrm{i} y_n)/\sqrt{2}$, the KdV Hamiltonian becomes

$$H_c = \sum_{n \geq 1} \lambda_n |q_n|^2 + \sum_{k+l+m=0} \gamma_k \gamma_l \gamma_m q_k q_l q_m$$

on $\hbar_{3/2}$ with

$$\lambda_n = (2\pi n)^3 + 6c \cdot 2\pi n.$$

Thus, at the origin we have an elliptic equilibrium with characteristic frequencies $\lambda_n, n \geq 1$.

To transform this Hamiltonian into its Birkhoff normal form up to order four two coordinate transformations are required: one to eliminate the cubic term, and one to normalize the resulting fourth order term. Both calculations are elementary. Expressed in real coordinates (x, y) the result is the following.

Theorem 5.10 *There exists a real analytic, symplectic coordinate transformation Φ in a neighbourhood of the origin in $\hbar_{3/2}$, which transforms the KdV Hamiltonian on $\hbar_{3/2}$ into*

$$H_c \circ \Phi = \frac{1}{2} \sum_{n \geq 1} \lambda_n (x_n^2 + y_n^2) - \frac{3}{4} \sum_{n \geq 1} (x_n^2 + y_n^2)^2 + \ldots,$$

where the dots stand for terms of higher order in x and y.

The important fact about the non-resonant Birkhoff normal form is that its coefficients are uniquely determined independently of the normalizing transformation, as long as it is of the form Òidentity + higher order termsÓ. For this reason, these coefficients are also called *Birkhoff invariants*. Comparing Theorem 5.10 with Theorem 2.1 and viewing Ψ as a *global* transformation into a *complete* Birkhoff normal form we thus conclude that the two resulting Hamiltonians on $\hbar_{3/2}$ must agree up to terms of order four. In other words, the local result provides us with the first terms of the Taylor series expansion of the globally integrable KdV Hamiltonian.

Corollary 5.11 *The canonical transformation Ψ of Theorem 2.1 transforms the KdV Hamiltonian into the Hamiltonian*

$$H_c(I) = \sum_{n \geq 1} \lambda_n I_n - 3 \sum_{n \geq 1} I_n^2 + \ldots,$$

where $I_n = \frac{1}{2}(x_n^2 + y_n^2)$, and the dots stand for higher order terms in (x, y). Consequently,

$$\omega_n(I) = \frac{\partial H_c}{\partial I_n}(I) = \lambda_n - 6 I_n + \ldots.$$

Here, λ_n and hence ω_n also depend on c.

By further computing some additional terms of order six in the expansion above, we gain sufficient control over the frequencies ω to verify all nondegeneracy and nonresonance conditions for any c.

Incidentally, the normal form of Theorem 5.10 already suffices to prove the persistence of quasi-periodic solutions of the KdV equation of sufficiently *small amplitude* under small Hamiltonian perturbations. In addition, if the perturbing term $\partial K/\partial u$ is of degree three or more in u, then no small parameter ε is needed to make the perturbing terms small, as it suffices to work in a sufficiently small neighbourhood of the equilibrium solution $u \equiv 0$. We will not expand on this point.

6 A Remark on the KAM Proof

Previous versions of the KAM theorem for partial differential equations such as [35, 54] were concerned with perturbations that were given by *bounded* nonlinear operators. This was sufficient to handle, among others, nonlinear Schrödinger and wave equations on a bounded interval, see for example [5, 39, 55]. This is *not* sufficient, however, to deal with perturbations of the KdV equation, as here the term

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial K}{\partial u}$$

is an *unbounded* operator. This entails some subtle difficulties in the proof of the KAM theorem, as we outline now.

Write the perturbed Hamiltonian as

$$H = N + P,$$

where N denotes some integrable normal form and P a general perturbation. The KAM proof employs a rapidly converging iteration scheme of Newton type to handle small divisor problems, and involves an infinite sequence of coordinate transformations. At each step a transformation Φ is constructed as the time-1-map $X_F^t|_{t=1}$ of a Hamiltonian vector field X_F that brings the perturbed Hamiltonian $H = N + P$ closer to some new normal form N_+. Its generating Hamiltonian F as well as the correction $\hat{N}$ to the given normal form N are a solution of the linearized equation

$$\{F,N\} + \hat{N} = R,$$

where R is some suitable truncation of the Taylor and Fourier expansion of P. Then Φ takes the truncated Hamiltonian $H' = N + R$ into $H' \circ \Phi = N_+ + R_+$, where $N_+ = N + \hat{N}$ is the new normal form and

$$R_+ = \int_0^1 \{(1-t)\hat{N} + tR,F\} \circ X_F^t \,\mathrm{d}t$$

the new error term arising from R. Accordingly, the full Hamiltonian $H = N + P$ is transformed into $H \circ \Phi = N_+ + R_+ + (P - R) \circ \Phi$.

What makes this scheme more complicated than previous ones is the fact that the vector field X_R generated by R represents an *unbounded* operator, whereas the vector field X_F generated by the solution F of the linearized equation has to represent a *bounded* operator to define a *bona fide* coordinate transformation. For most terms in F this presents no problem, because they are obtained from the corresponding terms in R by dividing with a *large divisor*. There is no such smoothing effect, however, for that part of R of the form

$$\frac{1}{2}\sum_{n\notin A} R_n(\theta;\xi)(x_n^2+y_n^2),$$

where $\theta=(\theta_n)_{n\in A}$ are the coordinates on the torus $\mathbb{T}^A$, and ξ the parameters mentioned above. We therefore include these terms in $\hat{N}$ and hence in the new normal form N_+. However, subsequently we have to deal with a generalized, *θ-dependent* normal form

$$N=\sum_{n\in A}\omega_n(\xi)I_n+\frac{1}{2}\sum_{n\notin A}\Omega_n(\theta;\xi)(x_n^2+y_n^2).$$

This, in turn, makes it difficult to obtain solutions of the linearized equation with useful estimates.

In [37] Kuksin obtained such estimates and thus rendered the iterative construction convergent. It requires a delicate discussion of a linear small divisor equation with *large, variable* coefficients.

7 Existing Literature

Theorem 2.1 was first given in [1] and [2]. A quite different approach to this result, and the one detailed here, was first presented in [26] and extended to the nonlinear Schrödinger equation in [23]. At the heart of the argument are orthogonality relations which first have been established in the case of the nonlinear Schrödinger equation by McKean & Vaninsky [47, 48].

A version of Theorem 4.2 in the case $c=0$ is due to Kuksin [34, 37]. In the second paper, he proves a KAM theorem of the type discussed above which is needed to deal with perturbations given by unbounded operators, and combines it with earlier results [3, 4] concerning nonresonance properties of the KdV frequencies and the construction of local coordinates so that the linearized equations of motions along a given torus of finite gap potentials reduce to constant coefficients [34].

The proof of Theorem 4.2 presented in [27] is different from the approach in [34, 37]. Instead of the local coordinates constructed in [34] we use the global, real analytic action-angle coordinates given by Theorem 2.1 to obtain quasi-periodic solutions of arbitrary size for sufficiently small and sufficiently regular perturbations of the KdV equation. To verify the relevant nonresonance conditions we follow the line of arguments used in [39] where small quasi-periodic solutions for nonlinear

Schrödinger equations were obtained, and explicitly compute the Birkhoff normal form of the KdV Hamiltonian up to order 4 and a few terms of order 6.

We stress again that our results are concerned exclusively with the existence of *quasi-periodic* solutions. Nothing is known about the persistence of almost-periodic solutions. The KAM theory of [53] concerning such solutions is not applicable here, since the nonlinearities effect a strong, long range coupling among all “modes” in the KdV equation.

There are, however, existence results for simplified problems. Bourgain [8, 9] considered the Schrödinger equation

$$\mathrm{i}u_t = u_{xx} - V(x)u - |u|^2 u$$

on $[0, \pi]$ with Dirichlet boundary conditions, depending on some analytic potential V. Given an almost-periodic solution of the linear equation with *very* rapidly decreasing amplitudes and nonresonant frequencies, he showed that the potential V may be modified so that this solution persists for the nonlinear equation. The potential serves as an infinite dimensional parameter, which has to be chosen properly for each initial choice of amplitudes. This result is obtained by iterating the Lyapunov-Schmidt reduction introduced by Craig & Wayne [11].

A similar result was obtained independently in [56] by iterating the KAM theorem about the existence of quasi-periodic solutions. As a result, one obtains for – in a suitable sense – almost all potentials V a set of almost periodic solutions, which – again in a suitable sense – has density one at the origin. See [56] for more details.

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