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Some Recent Results concerning Quasi-periodic Solutions for a Nonlinear String Equation

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1 Results

In this paper we are concerned with the nonlinear string equation

$$u_{tt} = u_{xx} - mu - f(u) \tag{1}$$

on the bounded interval $0 \le x \le \pi$ with Dirichlet boundary conditions

$$u(t, 0) = 0 = u(t, \pi), \qquad -\infty < t < \infty.$$

Here, $m > 0$ is a real parameter, sometimes referred to as the "mass", and f is a real analytic, odd function of u of the form

$$f(u) = au^3 + \sum_{k \ge 5} f_k u^k, \qquad a \ne 0. \tag{2}$$

This class of equations includes the sine-Gordon, the sinh-Gordon and the ϕ^4-equation, which are given by

$$mu + f(u) = \begin{cases} \sin u, \\ \sinh u, \\ u + u^3, \end{cases}$$

respectively, as well as odd perturbations of them of order five or more. After rescaling u they all reduce to the nonlinear string equation

$$u_{tt} = u_{tt} - mu \mp u^3 + O(u^5).$$

As we will see, the higher order terms may also depend on x.

It is well known that this equation can be written as an infinite dimensional hamiltonian system. As the phase space one may take, for example, the product of the usual Sobolev spaces $\mathcal{P} = H_0^1 \times L^2$ on $[0, \pi]$ with coordinates u and $v = u_t$. The hamiltonian is then

$$H = \tfrac{1}{2} \langle v, v \rangle + \tfrac{1}{2} \langle Au, u \rangle + \int_0^\pi g(u)\, \mathrm{d}x,$$

where $A = \mathrm{d}^2/\mathrm{d}x^2 + m$ and $g = \int_0 f(s)\mathrm{d}s$, and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in L^2. The hamiltonian equations of motions are

$$u_t = \frac{\partial H}{\partial v} = v, \quad v_t = -\frac{\partial H}{\partial u} = -Au - f(u), \tag{3}$$

hence they are equal to (1).

Our aim is to construct families of real analytic solutions u that are *quasi-periodic* in time and hence *a fortiori* globally defined. This means, that they can be written in the form

$$u(t, x) = U(\omega_1 t, \ldots, \omega_n t, x),$$

where U is a real analytic function of period 2π in all arguments except the last one, and $\omega_1, \ldots, \omega_n$ are rationally independent real numbers, the so called *basic frequencies* of the quasi-periodic function u. Thus, u admits a Fourier series expansion

$$u(t, x) = \sum_{k \in \mathbb{Z}^n} U_k(x) e^{\mathrm{i} k \cdot \omega t},$$

where $k \cdot \omega = \sum_j k_j \omega_j$. A special case are time periodic solutions, which are quasi-periodic with exactly one basic frequency.

From a geometric point of view, such solutions are given by *embeddings* of the n-torus $\mathbb{T}^n$ into the phase space $\mathcal{P}$,

$$\mathbb{T}^n \to \mathcal{P}, \quad \theta \mapsto (U(\theta, \cdot), DU(\theta, \cdot)),$$

where $DU = \sum_j \omega_j U_{\theta_j}$, such that the straight windings $t \mapsto \theta(t) = \omega t + \theta_0$ on the torus map into solutions of (3). In phase space we thus have embedded *invariant* *tori*, on which in suitable coordinates the flow is *linear*. We call them *rotational tori* in the sequel. Indeed, it will be these embeddings that we construct in the first place, while the initial value problem for (3) is never considered directly.

The solutions to be constructed are of *small amplitude*. Hence the higher order terms may be considered as a small perturbation of the linear terms, which amount to the Klein-Gordon equation $u_{tt} = u_{xx} - mu$, which has plenty of quasi-periodic solutions. Namely, let

$$\phi_j = \sqrt{\frac{2}{\pi}} \sin jx, \qquad \lambda_j = \sqrt{j^2 + m}$$

for $j \geq 1$ be its basic modes and their frequencies. Then every solution is the superposition of their harmonic oscillations, namely

$$u(t,x) = \sum_{j \geq 1} q_j(t)\phi_j(x), \qquad q_j(t) = I_j \cos(\lambda_j t + \varphi_j^\circ),$$

with amplitudes $I_j \geq 0$ and initial phases φ_j°. Their combined motion is periodic, quasi-periodic or almost periodic, respectively, depending on whether one, finitely many or infinitely many modes have positive amplitude. In particular, for every choice $J = \{ j_1 < j_2 < \cdots < j_n \} \subset \mathbb{N}$ of finitely many modes there is an invariant $2n$-dimensional linear subspace E_J that is completely foliated into rotational tori with fixed frequencies $\lambda_{j_1}, \ldots, \lambda_{j_n}$:

$$E_J = \left\{ (u,v) = (q_1\phi_{j_1} + \cdots + q_n\phi_{j_n}, p_1\phi_{j_1} + \cdots + p_n\phi_{j_n}) \right\} = \bigcup_{I \in \overline{P^n}} \mathcal{T}_J(I),$$

where $P^n = \left\{ I \in \mathbb{R}^n : I_j > 0 \text{ for } 1 \leq j \leq n \right\}$ is the positive quadrant in $\mathbb{R}^n$ and

$$\mathcal{T}_J(I) = \left\{ (u,v) : q_j^2 + \lambda_j^{-2} p_j^2 = I_j \text{ for } 1 \leq j \leq n \right\},$$

using the above representation of u and v. Each such torus is invariant, linearly stable, and all solutions have vanishing Lyapunov exponents. – This is the linear situation.

Upon restoring the nonlinearity f such an invariant manifold E_J will not persist in its entirety due to frequency-amplitude modulations and resonances among the modes. The main result, however, is that in a sufficiently small neighbourhood of the origin there does persist a large *Cantor subfamily* of rotational n-tori which are only slightly deformed. This means, there exists a Cantor set $\mathcal{C} \subset P^n$, parameterizing the family of n-tori

$$\mathcal{T}_J[\mathcal{C}] = \bigcup_{I \in \mathcal{C}} \mathcal{T}_J(I) \subset E_J$$

over $\mathcal{C}$, and a Lipschitz continuous embedding $\Phi\colon \mathcal{T}_J[\mathcal{C}] \to \mathcal{E}_J \subset \mathcal{P}$, such that the restriction of Φ to each $\mathcal{T}_J(I)$ in the family is an embedding of a rotational n-torus for the nonlinear equation. The image $\mathcal{E}_J$ of $\mathcal{T}_J[\mathcal{C}]$ we call a *Cantor manifold of rotational n-tori*.

These Cantor manifolds have a number of additional properties.

(1) In first approximation Φ agrees with the inclusion map $\Phi_0\colon E_J \hookrightarrow \mathcal{P}$ restricted to $\mathcal{T}_J[\mathcal{C}]$. Its restriction to each torus $\mathcal{T}_J(I)$ is real analytic, and it maps into the space of real analytic functions on $[0, \pi]$, with uniform domains of analyticity.

(2) The Cantor set $\mathcal{C}$ has full density at the origin:

$$\lim_{r \to 0} \frac{\mu(\mathcal{C} \cap B_r)}{\mu(P^n \cap B_r)} = 1,$$

where $B_r = \{I : \|I\| < r\}$ and μ denotes Lebesgue measure.

(3) $\mathcal{E}_J$ has a tangent space at the origin equal to E_J: $T_0\mathcal{E}_J = E_J$.

(4) The frequencies ω of the rotational tori are diophantine, whence we also call the latter *diophantine tori*. That is, there exist positive α and τ such that

$$|k \cdot \omega| \geq \frac{\alpha}{|k|^\tau}, \qquad 0 \neq k \in \mathbb{Z}^n.$$

The exponent τ can be kept fixed, while α tends to zero as the tori approach the origin.

(5) All the tori are linearly stable, and all their orbits have zero Lyapunov exponents.

The main result may be formulated as follows.

Main Theorem [12]. *Suppose the nonlinearity f is real analytic, of the form of (2) and odd: $f(-u) = -f(u)$. Then for each index set $J = \{j_1 < \cdots < j_n\}$ with $n \geq 2$ satisfying*

$$\min_{1 \leq i < n} j_{i+1} - j_i \leq n - 1$$

there exists for all $m > 0$ a Cantor manifold $\mathcal{E}_J$ of real analytic, linearly stable, diophantine n-tori for the nonlinear string equation given by a Lipschitz continuous embedding

$$\Phi\colon \ \mathcal{T}_J[\mathcal{C}] \to \mathcal{E}_J,$$

which in first approximation agrees with the inclusion map $\Phi_0\colon E_J \hookrightarrow \mathcal{P}$ restricted *to $\mathcal{T}_J[\mathcal{C}]$. The Cantor set $\mathcal{C}$ has full density at the origin, and $\mathcal{E}_J$ has a tangent space at the origin equal to E_J. Moreover, $\mathcal{E}_J$ is contained in the space of real analytic functions on $[0, \pi]$.*

For one point sets $J = \{j\}$ the same holds with the exception of at most finitely many m-values. In particular, there are no exceptions for $J = \{1\}$.

The assumption on $\min j_{i+1} - j_i$ is made to ensure that the Cantor manifolds exist for *all* positive m. Otherwise, one might have to exclude some set of m-values, which is discrete in every compact interval in $(0, \infty)$. But the condition given here is certainly not the sharpest. Also, we did not investigate the exceptional set for one point sets J thoroughly because for the existence of Cantor discs of *periodic* solutions there are better results [3]. It may well be, however, that there are no exceptional points at all.

The assumption that f is odd in u is necessary. The solutions constructed are real analytic sine-series, hence they are *odd* about $x = 0$. Addint the differential equations for $u(t, x)$ and $u(t, -x)$ one then obtains $f(u) + f(-u) = 0$. – Incidentally, this restriction does *not* arise with Neumann boundary conditions.

The size of the Cantor manifolds $\mathcal{E}_J$ is not uniform, but depends on m, n and J, and in particular tends to zero as n tends to infinity. Thus, unlike the linear spaces E_J, they are *not* dense in some fixed neighbourhood of the origin. But they are *asymptotically dense* in the following sense.

Corollary. *The union of all Cantor manifolds $\mathcal{E}_J$ intersects every nonempty open cone in $H_0^1 \times L^2$ with vertex at the origin.*

The above results are proven in a separate paper [12] and are based on the results of the joint work of S. Kuksin and the author [5,11]. In the following we give an outline of the proof, describe some extensions and indicate some open problems.

2 The KAM Approach

As already indicated, we are dealing with a perturbation problem for an infinite dimensional hamiltonian system. The aim is to continue finite dimensional invariant tori with quasi-periodic motions under the influence of an infinite dimensional perturbation. This calls for an extension of the well known KAM theory for finite dimensional almost integrable hamiltonian systems to infinite-dimensional systems. Such a theory, concerning finite dimensional tori, was recently developed mainly by Kuksin [4], and also by Wayne [15] and the author [11]. There is also a KAM theory for infinite-dimensional tori for some classes of infinite-dimensional hamiltonians [10], but these results are not applicable here.

To start, a suitable integrable system has to be found to apply the perturbation theory to. Interestingly, there are a number of possibilities for such a choice.

Linear System. The Klein-Gordon equation $u_{tt} = u_{xx} - mu$ with Dirichlet boundary conditions is integrable, and all its solutions are periodic, quasi-periodic or almost periodic. However, in a linear system there is no frequency-amplitude-modulation. For example, in the invariant spaces E_J the frequencies of the quasi-periodic solutions do not vary. Hence this system is completely *degenerate*, and KAM theory is *not* applicable.

The situation is different, if the scalar parameter m is replaced by some potential function $Q \in L^2([0,\pi])$. This amounts to introducing infinitely many parameters into the system, which may be adjusted and thus substitute the usual nondegeneracy condition. As a result one finds a *Cantor set of potentials* Q for which there are Cantor families of small amplitude quasi-periodic solutions. This approach was taken by Wayne [15]. However, that Cantor set surely does not include any open interval of constant potentials $Q \equiv m > 0$ due to *infinitely* many nonresonance conditions that have to be imposed on the frequencies λ_j.

Integrable PDE. The sine-Gordon equation and the sinh-Gordon equation with periodic boundary conditions are known to be integrable, exhibiting plenty of quasi-periodic solutions. They may serve as the starting point for a perturbation theory. This approach was taken by Bobenko and Kuksin [1], and essentially the same results were obtained. However, before KAM theory may be applied a formidable amount of work needs to be done to bring the equations into suitable form. This involves the use of hyperelliptic Riemann surfaces, theta-functions, Schottky uniformization, and other tools.

Integrable ODE. Here the starting point is the equation $u_{tt} = u_{xx} - mu \mp u^3$ with Dirichlet boundary conditions. This equation is *not* integrable. But by a single symplectic coordinate transformation, its hamiltonian is brought into *Birkhoff normal form* of order four with respect to any finite number of basic modes. Then KAM theory is applicable. This approach was suggested earlier by the author [9], and was carried out for the nonlinear Schrödinger equation by Kuksin and the author [5]. There some aspects are even simpler than here, such as the transformation into normal form and the checking of the relevant nondegeneracy conditions. In the following we describe the essential steps of this approach for the nonlinear string equation.

3 Outline of Proof

To start, we use the complete set of eigenfunctions ϕ_j of the operator A to write

$$u = \sum_{j\ge 1} \frac{q_j}{\sqrt{\lambda_j}} \phi_j, \qquad v = \sum_{j\ge 1} \sqrt{\lambda_j}\, p_j \phi_j .$$

The coordinates are taken from some Hilbert space $\ell^{a,s}$ of all real valued sequences $w = (w_1, w_2, \dots)$ with finite norm

$$\|w\|^2_{a,s} = \sum_{j\ge 1} |w_j|^2\, j^{2s} e^{2ja}$$

and $a > 0$, $s > \frac{1}{2}$. Thus these coefficients parameterize real analytic functions on $[0, \pi]$. We then obtain the hamiltonian

$$H = \Lambda + G = \tfrac{1}{2} \sum_{j\ge 1} \lambda_j (p_j^2 + q_j^2) + \int_0^\pi g(u(q))\,\mathrm{d}x \tag{4}$$

with equations of motions

$$\dot q_j = \frac{\partial H}{\partial p_j} = \lambda_j p_j, \qquad \dot p_j = -\frac{\partial H}{\partial q_j} = -\lambda_j q_j - \frac{\partial G}{\partial q_j}. \tag{5}$$

They are the hamiltonian equations of motion with respect to the standard symplectic structure $\sum \mathrm{d}q_j \wedge \mathrm{d}p_j$ on $\ell^{a,s} \times \ell^{a,s}$.

This hamiltonian may serve as our new starting point due to the following simple observation.

Lemma A. *If a curve* $I \to \ell^{a,s} \times \ell^{a,s}$, $t \mapsto (q(t), p(t))$ *is a real analytic solution of (5), then* $u(t,x) = \sum \lambda_j^{-1/2} q_j(t)\phi_j(x)$ *is a classical solution of (1) that is real analytic on* $I \times [0, \pi]$.

To continue our investigation we have to establish the regularity of the hamiltonian in the new coordinates.

Lemma B. *The gradient* G_q *is real analytic as a map from some neighbourhood of the origin in* $\ell^{a,s}$ *into* $\ell^{a,s+1}$, *and* $\|G_q\|_{a,s+1} = O\big(\|q\|^3_{a,s}\big)$.

Since G is independent of p, the associated hamiltonian vectorfield

$$X_G = \sum_{j\ge 1} \frac{\partial G}{\partial p_j} \frac{\partial}{\partial q_j} - \sum_{j\ge 1} \frac{\partial G}{\partial q_j} \frac{\partial}{\partial p_j}$$

defines a real analytic map from some neighbourhood of the origin in $\ell^{a,s} \times \ell^{a,s}$ into $\ell^{a,s+1} \times \ell^{a,s+1}$. Hence, X_G is *smoothing* of order 1. By contrast, the linear vectorfield X_Λ is *unbounded* of order 1.

For the nonlinearity u^3 we find

$$G = \tfrac{1}{4} \sum_{i,j,k,l} G_{ijkl}\, q_i q_j q_k q_l, \qquad G_{ijkl} = \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \int_0^\pi \phi_i \phi_j \phi_k \phi_l \,\mathrm{d}x. \tag{6}$$

One verifies that $G_{ijkl} = 0$ *unless* $i \pm j \pm k \pm l = 0$ for *some* combination of plus and minus signs. Thus, only a codimension one set of coefficients is actually different from zero, and the sum extends only over $i \pm j \pm k \pm l = 0$. In particular,

$$G_{iijj} = \frac{1}{2\pi} \frac{2 + \delta_{ij}}{\lambda_i \lambda_j}$$

by an elementary calculation.

Next we transform the hamiltonian of (4) into some partial Birkhoff normal form of order four so that it appears, in a sufficiently small neighbourhood of the origin, as a small perturbation of some *nonlinear integrable* system. To this end we introduce the usual complex coordinates

$$z_j = \tfrac{1}{\sqrt{2}}(q_j + \mathrm{i} p_j), \quad \bar{z}_j = \tfrac{1}{\sqrt{2}}(q_j - \mathrm{i} p_j).$$

We obtain a real analytic hamiltonian $H = \sum_j \lambda_j \, |z_j|^2 + \dots$ on the now *complex* Hilbert space $\ell_c^{a,s}$ with symplectic structure $\mathrm{i} \sum_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$. Real analytic means, that H is a function of z and $\bar{z}$, which is real analytic in the real and imaginary part of z. We formulate the result for the nonlinearity u^3, since higher order terms do not make a difference.

Lemma C. *For each finite $n \geq 1$ and each $m > 0$ there exists a real analytic, symplectic change of coordinates Γ is some neighbourhood of the origin in $\ell_c^{a,s}$ such that*

$$H \circ \Gamma = \sum_{j \geq 1} \lambda_j \, |z_j|^2 + \tfrac{1}{2} \sum_{\min(i,j) \leq n} \bar{G}_{ij} \, |z_i|^2 \, |z_j|^2 + O\big(\|\hat{z}\|_{a,s}^4\big) + O\big(\|z\|_{a,s}^6\big),$$

where $\hat{z} = (z_{n+1}, z_{n+2}, \dots)$ and $\bar{G}_{ij} = \frac{6}{\pi} \cdot \frac{4 - \delta_{ij}}{\lambda_i \lambda_j}$. Moreover, the regularity of the hamiltonian vectorfields is preserved.

In the new coordinates the interaction of the first n modes is thus normalized, and dropping the O-terms the hamiltonian becomes completely integrable. In a sufficiently small neighbourhood of the origin the latter can and will be considered as a small perturbation of this nonlinear integrable system.

The proof of the lemma employs the Lie transform method [6] in its simplest form to generate a single transformation that removes the bad terms of order four. The crucial observation is that the relevant divisors of order four are uniformly bounded away from zero. Namely, if $i \pm j \pm k \pm l = 0$, with no two pairs of integers of opposite sign, then

$$\left|\lambda_i \pm \lambda_j \pm \lambda_k \pm \lambda_l\right| \geq \frac{cm}{\sqrt{n^2+m}^3}, \qquad n = \min(i, j, k, l), \tag{7}$$

with some absolute constant c. On the other hand, we can not transform away the $\hat{z}$-term by an analytic transformation of the same kind because here the divisors tend to zero.

In a neighbourhood of the origin in $\ell_c^{a,s}$ we now consider more generally hamiltonians of the form $H = \Lambda + Q + R$, where $\Lambda + Q$ is integrable and in normal form and R is a perturbation term. More precisely, letting $z = (\tilde{z}, \hat{z})$ with $\tilde{z} = (z_1, \ldots, z_n)$, $\hat{z} = (z_{n+1}, z_{n+2}, \ldots)$, and furthermore $I = \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2)$, $Z = \frac{1}{2}(|z_{n+1}|^2, |z_{n+2}|^2, \ldots)$, we assume that

$$\Lambda = \langle \alpha, I \rangle + \langle \beta, Z \rangle, \qquad Q = \tfrac{1}{2}\langle AI, I \rangle + \langle BI, Z \rangle,$$

with constant vectors α, β and constant matrices A, B. In the Birkhoff normal form lemma the first two terms are of that form.

The equations of motion of the hamiltonian $\Lambda + Q$ are

$$\dot{\tilde{z}}_i = \mathrm{i}(\alpha + AI + B^T z)_i \tilde{z}_i, \qquad \dot{\hat{z}}_j = \mathrm{i}(\beta + BI)_j \hat{z}_j.$$

Thus, the complex n-dimensional manifold $E = \{\hat{z} = 0\}$ is invariant, and it is completely filled by invariant tori:

$$E = \bigcup_{I \in \overline{P^n}} \mathcal{T}(I), \qquad \mathcal{T}(I) = \left\{\tilde{z} : |\tilde{z}_j|^2 = 2I_j, 1 \leq j \leq n\right\}.$$

On $\mathcal{T}(I)$ and in its normal space, respectively, the flows are given by

$$\dot{\tilde{z}}_i = \mathrm{i}\omega_i(I)\tilde{z}_i, \qquad \omega(I) = \alpha + AI,$$
$$\dot{\hat{z}}_j = \mathrm{i}\Omega_j(I)\hat{z}_j, \qquad \Omega(I) = \beta + BI.$$

They are linear and in diagonal form. In particular, since $\Omega(I)$ is real, $\hat{z} = 0$ is an elliptic fixed point, all the tori are linearly stable, and all their orbits have zero Lyapunov exponents. Thus, $\mathcal{T}(I)$ is an *elliptic rotational torus with frequencies* $\omega(I)$.

Note that although looking very much alike there is an important difference between an n-dimensional space E_J for the linear Klein-Gordon equation and the space E for the nonlinear normal form hamiltonian. In the latter the frequencies in general vary from torus to torus, while in the former they do not. This is essential for the stability result below.

Including the nonintegrable perturbation term R this manifold E will not persist in its entirety due to resonances among the oscillations. We show, however, the persistence of a *large portion* of E forming an invariant Cantor manifold $\mathcal{E}$ for the hamiltonian $H = \Lambda + Q + R$.

For the existence of $\mathcal{E}$ the following assumptions are made.

A. Nondegeneracy. The normal form $\Lambda + Q$ is *nondegenerate* in the sense that

$$\begin{aligned} &(\mathrm{A}_1) && \det A \neq 0, \\ &(\mathrm{A}_2) && \langle l, \beta \rangle \neq 0, \\ &(\mathrm{A}_3) && \langle k, \omega(I) \rangle + \langle l, \Omega(I) \rangle \not\equiv 0, \end{aligned}$$

for all $(k, l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty$ with $1 \leq |l| \leq 2$.

B. Spectral Asymptotics. There exists $d \geq 1$ and $\delta < d - 1$ such that

$$\beta_j = j^d + \cdots + O(j^\delta),$$

where the dots stand for terms of order less than d in j.

C. Regularity. The hamiltonian vectorfields X_Q and X_R are real analytic maps from some neighbourhood of the origin in $\ell_c^{a,s}$ into $\ell_c^{a,\bar{s}}$, where $\bar{s} \geq s$, and in particular $\bar{s} > s$ for $d = 1$.

The regularity assumption for X_Q implies that for $d = 1$ there exists a maximal positive exponent κ such that

$$\frac{\Omega_i - \Omega_j}{i - j} = 1 + O(j^{-\kappa}), \qquad i > j,$$

uniformly for bounded I. For $d > 1$, we set $\kappa = \infty$.

The following theorem is formulated so that it applies, for example, also to a nonlinear Schrödinger equation.

The Cantor Manifold Theorem [5,11]. *Suppose the hamiltonian $H = \Lambda + Q + R$ satisfies assumptions A, B and C, and*

$$|R| = O\big(\|\hat{z}\|_{a,s}^4\big) + O\big(\|z\|_{a,s}^g\big)$$

with

$$g > 4 + \frac{4-\Delta}{\kappa}, \qquad \Delta = \min(\bar{s} - s, 1).$$

Then there exists a Cantor manifold $\mathcal{E}$ of real analytic, elliptic diophantine n-tori given by a Lipschitz continuous embedding $\Psi\colon\ \mathcal{T}[\mathcal{C}] \to \mathcal{E}$, where $\mathcal{C}$ has full density at the origin, and Ψ is close to the inclusion map Ψ_0: $\|\Psi - \Psi_0\|_{a,\bar{s},B_r\cap\mathcal{T}[\mathcal{C}]} = O(r^\sigma)$ with some $\sigma > 1$. Consequently, $\mathcal{E}$ is tangent to E at the origin.

In applying this theorem to the nonlinear string equation one first renumbers a finite number of modes so that the ones with index in $J = \{j_1 < \cdots < j_n\}$ become the first n modes. For the frequencies we then have

$$\beta_j = \lambda_{j+n} = j + n + \frac{m}{2j} + O(j^{-2}),$$

thus $d = 1$ and $\delta = -1$. Moreover, $\bar{s} = s + 1$ and $\kappa = 2$. This leads to the condition $g \geq 6$, whence the assumption that the nonlinearity f has no term of order four. This restriction, however, can be removed, as we will indicate below.

The other assumptions are verified easily except for assumption A_3. It is here that we need to put some restriction on the index set J in order to satisfy this condition for *all* $m > 0$. Once the hypotheses are verified the Main Theorem follows from the Cantor Manifold Theorem in a straightforward manner. As mentioned before, all these statements are proven elsewhere [12].

4 Further Remarks and Open Problems

The preceding results hinge on the presence of the term u^3 in the nonlinearity, irrespective of its sign, as this term determines the fourth order coefficients of the Birkhoff normal form. Higher order terms do not make any difference so long as they are at least of order five to contribute to the perturbation.

Therefore, the results remain true for odd nonlinearities f of the form

$$f(x,u) = au^3 + \sum_{k\geq 5} f_k(x)u^k, \qquad a \neq 0,$$

where the coefficients f_k are real analytic in x. Indeed, they may also belong to some Sobolev space $H^s([0,\pi])$, $s > \frac{1}{2}$, with norms growing at most exponentially to ensure analyticity in u. In the latter, non-analytic case the resulting quasi-periodic solutions are of class H^{s+2} in x only.

One may also add a general odd perturbation term

$$\epsilon g(x,u) = \epsilon \sum_{k\geq 0} g_k(x) u^k$$

to the nonlinearity f, with coefficients g_k of the same type as the f_k. Then there still exist Cantor manifolds for all sufficiently small ϵ, the smallness depending on m, n and J. However, they are not dense at the origin, but have a *hole* there, since the perturbation no longer tends to zero as we approach the origin.

Indeed, one may also allow for terms of order four of the same kind. This requires an extension of the underlying basic KAM theorem [11] along the lines of Kuksin's approach [4], which is explained there. We do not give a formal statement of this extension, however, because this slightly greater generality seems to be of minor importance as far as this string equation is concerned.

The coefficient of u^3, on the other hand, must be *independent* of x for our results to hold. Otherwise, one has $G_{ijkl} \neq 0$ in (6) for practically *all* indices, and there would be no uniform lower bound for the relevant divisors in (7). There still is a normalizing transformation, but the new hamiltonian vectorfield looses its smoothing property.

For the same reason, we have to exclude a quadratic term u^2. The corresponding contribution of order three to the hamiltonian may be transformed away, but only at the expense of loosing the smoothing property, which is required by assumption C of the Cantor Manifold Theorem. It is an open question whether these restrictions are a shortcoming of our technique, or whether they are related to non-quasi-periodic dynamical behavior of the system.

Many of the difficulties and restrictions in the construction of quasi-periodic solutions along these lines are due to the appearance of a certain kind of small divisors, namely $\langle k,\omega\rangle + \Omega_i - \Omega_j$ for $i \neq j$ and integer vectors k, *including* $k = 0$, for which we need a diophantine estimate of the form

$$\left|\langle k,\omega\rangle + \Omega_i - \Omega_j\right| \geq c\,\frac{\left|i^d - j^d\right|}{1+|k|^\tau}.$$

This, for example, excludes asymptotically double frequencies, whence we can not handle the case of periodic boundary conditions so far. These divisors enter the proof because the KAM method employs successive coordinate transformations to put the whole set of equations of motions into a "normal form". This requires some control over the variational equations along the torus to be constructed, which involves the above divisors [7].

It is an open question, whether they should be significant, when one is only interested in the invariant torus, but not in the variational equations along it. If one were able just to construct the *embedding* of such a torus, without resorting to coordinate transformations, then these divisors might be avoided, leading to more general results.

This is for example the case with the classical Lyapunov center theorem concerning the existence of small discs of periodic solutions around an equilibrium [14]. Another example is the construction of lower dimensional invariant manifolds for a holomorphic map near an elliptic fixed point, on which the map is linearizable [8]. This is an extension of Siegel's famous center theorem [13], which concerns the linearization of a conformal map at an elliptic fixed point. In this particular case one can show that a KAM type proof requires extra small divisor conditions which are not necessary [8].

Now, in a recent pioneering paper Craig and Wayne [3] extended the Lyapunov center theorem to the infinite-dimensional hamiltonian of the nonlinear string equation, obtaining *Cantor discs* of *periodic* solutions. They were able to solve the functional equation for the *embedding* of such a disc by a Lyapunov Schmidt reduction and a Newton iteration scheme which also required the use of KAM type estimates. As a result, they can allow for periodic boundary conditions as well as more general nonlinearities. Thus, their results are more general then the Main Theorem above for the case $n = 1$. It remains to be seen whether their approach can be extended to *quasi-periodic* solutions.

Another open question concerns higher dimensional x-domains. By condition B, the underlying linear pde has to have a pure point spectrum with eigenvalues λ_j which grow at least linearly with j. This restricts the applicability of the Cantor Manifold Theorem essentially to one-dimensional problems. Again, the technical reason behind this is the necessity to control the divisors $\langle k, \omega \rangle + \Omega_i - \Omega_j$ on sets of positive measure. It is therefore not clear whether the results extend to higher dimensional domains.

The existence of *almost periodic* solutions is yet another open question. The unperturbed systems are full of them. But the perturbation theories for almost periodic solutions on infinite-dimensional tori available so far are clearly not applicable to the hamiltonians of pde [2,10]

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