Version 1.0, June 2007

Spectral Gaps of Potentials in Weighted Sobolev Spaces

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1 Results

We consider the Schrödinger operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q$$

on the interval [0, 1] depending on an L^2-potential q and endowed with periodic or anti-periodic boundary conditions. In this case, L is also known as *Hill's operator*. Its spectrum is pure point, and for real q consists of an unbounded sequence of real *periodic eigenvalues*

$$\lambda_0^+(q) < \lambda_1^-(q) \leqslant \lambda_1^+(q) < \cdots < \lambda_n^-(q) \leqslant \lambda_n^+(q) < \cdots \ .$$

Their asymptotic behaviour is

$$\lambda_n^{\pm} = n^2\pi^2 + [q] + \ell^2(n),$$

where $[q]$ denotes the mean value of q and $\ell^2(n)$ a generic square sumable term. Equality may occur in every place with a '$\leqslant$'-sign, and one speaks of the *gap lengths*

$$\gamma_n(q) = \lambda_n^+(q) - \lambda_n^-(q), \qquad n \geqslant 1,$$

of the potential q.

For complex q, the periodic eigenvalues are still well defined, but in general not real, since L is no longer self-adjoint. Their asymptotic behaviour is the same, however, and we may order them lexicographically – first by their real part, then by their imaginary part – so that

$$\lambda_0(q) \prec \lambda_1^-(q) \preccurlyeq \lambda_1^+(q) \prec \cdots \prec \lambda_n^-(q) \preccurlyeq \lambda_n^+(q) \prec \cdots .$$

The gap lengths are then defined as before, but are now complex valued in general.

Classical Results

We are interested in the relationship between the regularity of a potential and the sequence of its gap lengths. Hochstadt [5] observed that

$$q \in C^\infty(S^1, \mathbb{R}) \quad \Leftrightarrow \quad \gamma_n(q) = O(n^{-k}) \text{ for all } k \geqslant 0,$$

and Marčenko & Ostrowskĭ [8] subsequently showed that

$$q \in H^m(S^1, \mathbb{R}) \quad \Leftrightarrow \quad \sum_{n \geqslant 1} n^{2m} \gamma_n^2(q) < \infty$$

for all nonnegative integers m. Trubowitz [12] then proved that

$$q \in C^\omega(S^1, \mathbb{R}) \quad \Leftrightarrow \quad \gamma_n(q) = O(\mathrm{e}^{-an}) \text{ for some } a > 0.$$

Later, due to the realization of the periodic KdV flow as an isospectral deformation of Hill's operator, other regularity classes such as Gevrey functions and non-real potentials came into focus. Recent results in this direction appear example in [1, 2, 6, 7, 11]. Within certain limits, one may think of the gap lengths as another kind of Fourier coefficients of the potential.

It is the purpose of this note to describe some of these recent developments. For more detailed statements and proofs we refer to [10] and later [3].

Weighted Sobolev spaces

As in [6, 7] a *weight* is a *normalized*, *symmetric* and *submultiplicative* function $w\colon \mathbb{Z} \to \mathbb{R}$. That is, for all integers n and m, we have

$$w_n \geqslant 1, \qquad w_{-n} = w_n, \qquad w_{n+m} \leqslant w_n w_m.$$

Within the standard Sobolev space

$$\mathcal{H}^0 = L^2(S^1, \mathbb{C})$$

of square-integrable functions $u = \sum_{n\in\mathbb{Z}} u_n \mathrm{e}^{2\pi i n x}$ we then define the *weighted Sobolev spaces*

$$\mathcal{H}^w = \left\{ u \in \mathcal{H}^0 : \|u\|_w^2 := \sum_{n\in\mathbb{Z}} w_n^2 \, |u_n|^2 < \infty \right\}.$$

To give some examples, let $\langle n \rangle = 1 + |n|$. The *Sobolev weights* $\langle n \rangle^r$, $r \geqslant 0$, give rise to the usual Sobolev spaces $\mathcal{H}^r$ of 1-periodic, complex-valued functions. The *Abel weights*[1] $\langle n \rangle^r \mathrm{e}^{a|n|}$ with $a > 0$ define spaces $\mathcal{H}^{r,a}$ of functions in $\mathcal{H}^r$, which are analytic on the complex strip $|\mathrm{Im}\, z| < a/2\pi$ and have traces in $\mathcal{H}^r$ on the boundary lines. The *Gevrey weights*

$$w_n = \langle n \rangle^r \mathrm{e}^{a|n|^\sigma}, \qquad r \geqslant 0,\ a > 0,\ 0 < \sigma < 1,$$

lie in between and give rise to the so called Gevrey spaces $\mathcal{H}^{r,a,\sigma}$ of smooth functions. Obviously,

$$\mathcal{H}^{r,a} = \mathcal{H}^{r,a,1} \subsetneq \mathcal{H}^{r,a,\sigma} \subsetneq \mathcal{H}^{r,a,0} = \mathcal{H}^r.$$

Yet another weight is for example

$$w_n = \langle n \rangle^r \exp\left(\frac{a\,|n|}{1 + \log^\alpha \langle n \rangle} \right), \qquad \alpha > 0.$$

Since $\log w_n$ is subadditive and nonnegative, the limit

$$\chi(w) := \lim_{n\to\infty} \frac{\log w_n}{n}$$

exists and is nonnegative [9, no. 98]. Naturally, we call a weight w *exponential*, if $\chi(w) > 0$. We call w *subexponential*, if $\chi(w) = 0$ *with* $n^{-1} \log w_n$ converging to zero in an eventually monotone manner. This is not an exact dichotomy, but we are not aware of any interesting weight that does not belong to either class. Clearly, Abel weights are exponential, while Sobolev and Gevrey weights are subexponential.

Theorems

Let

[1] The term *Abel* weights is chosen to go along with *Sobolev* and *Gevrey* weights and has no deeper meaning.

$$\hbar^w = \left\{ u = (u_n)_{n\geqslant 1} \in \ell^2 : \sum_{n\geqslant 1} w_n^2 \left|u_n\right|^2 < \infty \right\},$$

and $\gamma(q) = (\gamma_n(q))_{n\geqslant 1}$.

Theorem 1 *If $q \in \mathcal{H}^w$, then $\gamma(q) \in \hbar^w$. In particular,*

$$\sum_{n\geqslant N} w_n^2 \left|\gamma_n(q)\right|^2 \leqslant 9 \left\|T_N q\right\|_w^2 + \frac{288}{N} \left\|q\right\|_w^4$$

for all $N \geqslant 4 \left\|q\right\|_w$, where $T_N q = \sum_{|n|\geqslant N} q_n \mathrm{e}^{2n\pi i x}$.

There is no one-to-one converse to Theorem 1 for exponential weights, as there exist real analytic finite gap potentials such as the Weierstrass $\wp$-function, which are not entire functions. In this case, fixing any $r > 0$, we have $(\gamma_n(u)) \in \hbar^{r,a}$ for *all* $a > 0$, but *only* $u \in \mathcal{H}^{r,a}$ for $a < a_0$. There is, however, a true converse for subexponential weights. We first consider the real case.

Theorem 2 *Suppose $q \in \mathcal{H}^0$ is real and $\gamma(q) \in \hbar^w$. If w is subexponential, then $q \in \mathcal{H}^w$. If w is exponential, then q is real analytic.*

Corollary 3 *If q is real and w is subexponential, then*

$$q \in \mathcal{H}^w \quad \Leftrightarrow \quad \gamma(q) \in \hbar^w.$$

For complex potentials, the spectral gap lengths alone do not suffice to determine the regularity of a potential. For instance, Gasymov [4] showed that *all* gap lengths vanish for complex potentials of the form

$$q = \sum_{n\geqslant 1} q_n \mathrm{e}^{2n\pi i x} = \sum_{n\geqslant 1} q_n z^n \Big|_{z=\mathrm{e}^{2\pi i x}}.$$

But Sansuc & Tkachenko [11] noted that the situation can be remedied by taking into account additional spectral data. In particular, they considered the quantities

$$\delta_n = \mu_n - \tau_n = \mu_n - (\lambda_n^+ - \lambda_n^-)/2,$$

where μ_n denotes the n-th Dirichlet eigenvalue. The quantities

$$\Gamma_n = \left|\gamma_n\right| + \left|\delta_n\right|,$$

may be considered as a measure of the size of the *spectral triangle* formed by the points λ_n^-, δ_n and λ_n^+. Note that $\gamma_n \leqslant \Gamma_n \leqslant 2\gamma_n$ for real potentials.

We then have the following converse theorem.

Theorem 4 *Suppose $q \in \mathcal{H}^0$ is real or complex and $\Gamma(q) \in h^w$. If w is subexponential, then $q \in \mathcal{H}^w$. If w is exponential, then q is real analytic.*

Corollary 5 *If w is subexponential, then*

$$q \in \mathcal{H}^w \quad \Leftrightarrow \quad \Gamma(q) \in h^w.$$

For the sake of brevity and simplicity, we will describe the line of reasoning for the real case. For more details, complete proofs and the complex case we refer to [10] and also [3].

2 Reduction

The idea of the proof of Theorem 1 is due to Kappeler & Mityagin [7]. They employ a Lyapunov-Schmidt reduction scheme called *Fourier block decomposition*.

The aim is to determine those λ near $n^2\pi^2$ with n sufficiently large, for which the equation $-y'' + qy = \lambda y$ admits a nontrivial 2-periodic solution f. As q can be considered small for large n, one expects its dominant modes to be $\mathrm{e}^{\pm n\pi ix}$. So it makes sense to separate these modes from the other ones by a Lyapunov-Schmidt reduction.

To this end we consider a similarly defined space $\mathcal{H}^w_\star$ of *2-periodic* functions, and write

$$\begin{aligned}\mathcal{H}^w_\star &= \mathcal{P}_n \oplus \mathcal{Q}_n \\ &= \operatorname{span}\{e_n, e_{-n}\} \oplus \operatorname{span}\{e_k : |k| \neq n\},\end{aligned}$$

where $e_k = \mathrm{e}^{k\pi ix}$. The projections onto $\mathcal{P}_n$ and $\mathcal{Q}_n$ are denoted by P_n and Q_n, respectively. With

$$f = u + v = P_n f + Q_n f,$$

Hill's equation decomposes into the so called P- and Q- equations

$$\begin{aligned}A_\lambda u &= P_n V(u + v), \\ A_\lambda v &= Q_n V(u + v),\end{aligned}$$

where $A_\lambda f = f'' + \lambda f$ and $Vf = qf$.

The operator A_λ has a compact inverse on $\mathcal{Q}_n$, when λ is near $n^2\pi^2$. Indeed, this holds on the complex strips $U_n = \{\lambda : |\operatorname{Re}\lambda - n^2\pi^2| \leqslant 12n\}$ for $n \geqslant 1$.

Lemma 1 *For $q \in \mathcal{H}^w$ and $\lambda \in U_n$, the operator $T_n = VA_\lambda^{-1}Q_n$ exists and is bounded on $\mathcal{H}^w_\star$ with norm*

$$\|T_n\|_w \leqslant \frac{2}{n}\,\|q\|_w\,.$$

Left-multiplying the Q-equation with VA_λ^{-1} we obtain $Vv = T_nVu + T_nVv$. For n large enough, T_n is a contraction on $\mathcal{H}^w_\star$, and there is a unique solution

$$Vv = \hat{T}_nT_nVu, \qquad \hat{T}_n = (I - T_n)^{-1}.$$

Inserted into the P-equation we get $A_\lambda u = P_nVu + P_n\hat{T}_nT_nVu = P_n\hat{T}_nVu$. So the P- and Q-equation reduce to the S-equation

$$S_nu = 0, \qquad S_n = A_\lambda - P_n\hat{T}_nV.$$

Since $\mathcal{P}_n$ is two-dimensional with basis e_n, e_{-n}, we have the matrix representation

$$S_n = \begin{pmatrix} \lambda - n^2\pi^2 - a_n & -c_n \\ -c_{-n} & \lambda - n^2\pi^2 - a_n \end{pmatrix},$$

with

$$a_n = \langle \hat{T}_nVe_n, e_n\rangle\,, \qquad c_n = \langle \hat{T}_nVe_{-n}, e_n\rangle\,.$$

Any nontrivial solution u gives rise to a 2-periodic solution of $A_\lambda f = Vf$, and vice versa. Hence the following holds.

Lemma 2 *A complex number λ near $n^2\pi^2$ is a periodic eigenvalue of q if and only if the determinant of S_n vanishes.*

Moreover, from the representations of a_n and c_n one easily obtains the following facts, which we need later.

Lemma 3 *For $n \geqslant 4\,\|q\|_w$ and $\lambda \in U_n$,*

$$|a_n - q_0|_{U_n}\,,\ w_n\,|c_n - q_n|_{U_n}\,,\ w_n\,|c_{-n} - q_{-n}|_{U_n} \leqslant \frac{4}{n}\,\|q\|_w^2\,.$$

Moreover, these coefficients are analytic functions of λ and q.

3 Gap Estimates

With the preceding results the forward problem of estimating the gap lengths of a potential is fairly straightforward. The determinant of S_n is the quadratic polynomial

$$\det S_n = (\lambda - n^2\pi^2 - a_n)^2 - |c_n|^2 ,$$

and the distance of its two roots has to be of the order of $|c_n|$.

Lemma 4 *For $n \geqslant 4\,\|q\|_w$ the determinant of S_n has exactly two roots $\xi_n^\pm$ in U_n, which are contained in the disc $|\lambda - n^2\pi^2| \leqslant 6\,\|q\|_w$ and satisfy*

$$\left|\xi_n^+ - \xi_n^-\right|^2 \leqslant 9\,|c_n c_{-n}|_{U_n} .$$

A counting argument then shows that these two roots have to be the two eigenvalues $\lambda_n^\pm$. Consequently, we obtain

$$|\gamma_n|^2 = \left|\xi_n^+ - \xi_n^-\right|^2 \leqslant 9\,|c_n c_{-n}|_{U_n} \leqslant 9\,|q_n|^2 + 9\,|q_{-n}|^2 + \frac{144}{n^2 w_n^2}\,\|q\|_w^4$$

by Lemma 3. Multiplying by w_n^2 and summing over $n \geqslant N$ we obtain Theorem 1.

4 Coefficient Estimates

We now turn to the more subtle problem of estimating the asymptotic behaviour of the Fourier coefficients of a potential in terms of its gap lengths. The geometric aspect is rather straightforward, at least in the real case. The off diagonal elements of S_n have to be bounded in terms of the gap lengths, that is

$$|c_n| \lessdot |\gamma_n| , \qquad n \gg 1,$$

where the dot stands for some implicit constant. The identity $c_n = \langle \hat{T}_n V e_{-n}, e_n \rangle$ then leads to an infinite dimensional system of nonlinear equations

$$c_n = q_n + O_2(\dots, q_k, \dots), \qquad n \neq 0,$$

which allows us to bound the q_n in terms of the c_n and hence in terms of the γ_n. See [1] for the lengthy details of this argument.

Here we present a functional analytic approach based on the fact that the coefficients of S_n – which contain all the necessary data – are analytic functions of q. Indeed, it even suffices to consider S_n at that value of λ where its diagonal vanishes.

For $m \geqslant 1$ and any weight w we introduce the centered balls

$$B_m^w = \{q \in \mathcal{H}^w : \|q\|_w \leqslant m/4\} \subset \mathcal{H}^0.$$

We also assume from now on that q has zero mean, that is, $\int_0^1 q \,\mathrm{d}x = q_0 = 0$, since adding a constant to q shifts its entire spectrum, but does not affect its gap lengths.

Using a contraction argument it is easy to show that the diagonal of S_n vanishes at a unique point α_n near $n^2\pi^2$.

Lemma 5 *For $m \geqslant 1$ and $n \geqslant m$ there exists a unique real analytic function*

$$\alpha_n : \; B_m^0 \to \mathbb{C}, \qquad \left|\alpha_n - n^2\pi^2\right|_{B_m^0} \leqslant \frac{m^2}{4n},$$

such that $\lambda - n^2\pi^2 - a_n(\lambda, \cdot)\big|_{\lambda=\alpha_n} \equiv 0$ *on* B_m^0.

Given $q \in \mathcal{H}^0$ we replace its Fourier coefficients q_n for $|n|$ large enough by

$$p_n = c_n(\alpha_n(q), q) = q_n + \dots \; .$$

The point is that

$$S_n(\alpha_n, q) = \begin{pmatrix} 0 & -p_n \\ -p_{-n} & 0 \end{pmatrix},$$

so these Fourier coefficients are well adapted to the lengths of the correponding gaps. More precisely, we define a map $\Phi_m : \; B_m^0 \to \mathcal{H}^0$ by

$$\Phi_m(q) = \sum_{|n|<M_m} q_n e_{2n} + \sum_{|n|\geqslant M_m} c_n(\alpha_n(q), q) e_{2n},$$

where, say, $M_m = 2^{10} m^2$. This is a near identity map with the following properties.

Proposition 6 *For each $m \geqslant 1$ and every weight w, the restriction of Φ_m to B_m^w is a real analytic diffeomorphism*

$$\Phi_m : \; B_m^w \to \Phi_m(B_m^w) \subset \mathcal{H}^w,$$

such that $\|D\Phi_m - I\|_{B_m^w} \leqslant 1/8$ *and*

$$2^{-1}\|q\|_w \leqslant \|\Phi_m(q)\|_w \leqslant 2\|q\|_w\,, \qquad q \in B_m^w.$$

By the last token, the image of B_m^w under Φ_m covers the ball $B_{m/2}^w$. Hence we have the following »abstract regularity result«.

Proposition 7 *If $q \in B_m^0$ for some $m \geqslant 1$ and $\Phi_m(q) \in B_{m/2}^w$ for some weight w, then $q \in B_m^w \subset \mathcal{H}^w$.*

Thus we would like to argue as follows. Given $q \in \mathcal{H}^0$ with a certain asymptotic behaviour of its gap lengths γ_n, we know that

$$\gamma_n \asymp c_{|n|} \asymp p_{|n|}, \qquad n \gg 1.$$

Choosing $m \asymp \|q\|_0$ so that $q \in B_m^0$, we thus have $p = \Phi_m(q) \in \mathcal{H}^w$. *If* we also had

$$(*) \qquad \|p\|_w \leqslant m/2,$$

then $\Phi_m(q) \in B_{m/2}^w$ and thus

$$q = \Phi_m^{-1}(p) \in \mathcal{H}^w,$$

by the preceding proposition. – Of course, given some $p \in \mathcal{H}^w$ there is no reason to have $\|p\|_w \leqslant m/2$. Simply increasing m does not help, since p depends on m.

5 Modified Weights

The idea is to *modify* the weight w in such a way that its asymptotics are preserved, while the norm $\|p\|_w$ is brought close to the norm $\|p\|_0$. For this to work flawlessly, however, we have to assume that w is *subexponential*.

So let $p \in \mathcal{H}^w$. Choosing m appropriately, we may assume that

$$0 < \|p\|_0 < m/6, \qquad \|p\|_w < \infty.$$

For $\varepsilon > 0$ we define a new function w_ε by

$$(w_\varepsilon)_n = \min(w_n, \mathrm{e}^{\varepsilon|n|}).$$

This is indeed a normalized, symmetric and submultiplicative function on $\mathbb{Z}$, hence a *weight*. Moreover, if w is subexponential, then clearly

$$(w_\varepsilon)_n = w_n, \qquad n \gg 1,$$

for any $\varepsilon > 0$.

If we now choose first N sufficiently large, and then ε sufficiently small, we can arrange that

$$\|T_N p\|_{w_\varepsilon} \leqslant \|T_N p\|_w \leqslant \|p\|_0 ,$$
$$\|p - T_N p\|_{w_\varepsilon} \leqslant 2\,\|p\|_0 ,$$

for $T_N p = \sum_{|n|\geqslant N} p_n e_{2n}$. Altogether, we have

$$\|p\|_{w_\varepsilon} \leqslant 3\,\|p\|_0 \leqslant m/2.$$

According to Proposition 7 we thus have $q = \Phi_m^{-1}(p) \in \mathcal{H}^{w_\varepsilon}$. But since w_ε has the same asymptotics as w we indeed have

$$q = \Phi_m^{-1}(p) \in \mathcal{H}^{w_\varepsilon} = \mathcal{H}^w,$$

as we wanted to show.

Essentially the same reasoning applies in the *exponential* case, with one important difference. If w is exponential, then

$$(w_\varepsilon)_n = \mathrm{e}^{\varepsilon|n|}, \qquad n \gg 1.$$

We thus may only conclude that

$$q = \Phi_m^{-1}(p) \in \mathcal{H}^{w_\varepsilon} = \mathcal{H}^{0,\varepsilon} \supsetneq \mathcal{H}^w.$$

So we conclude that q is real analytic, but its width of analyticity may be smaller than what the weight w may suggest.

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