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Hill's Potentials in Weighted Sobolev Spaces and their Spectral Gaps

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1 Results

In this paper we consider the Schrödinger operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q$$

on the interval $[0, 1]$, depending on an L^2-potential q and endowed with periodic or anti-periodic boundary conditions. In this case, L is also known as *Hill's operator*. Its spectrum is pure point, and for real q consists of an unbounded sequence of real *periodic eigenvalues*

$$\lambda_0^+(q) < \lambda_1^-(q) \leqslant \lambda_1^+(q) < \cdots < \lambda_n^-(q) \leqslant \lambda_n^+(q) < \cdots \ .$$

Their asymptotic behavior is

$$\lambda_n^\pm = n^2\pi^2 + [q] + \ell^2(n),$$

where $[q]$ denotes the mean value of q, and $\ell^2(n)$ a generic square summable term. Equality may occur in every place with a '$\leqslant$'-sign, and one speaks of the *gap lengths*

$$\gamma_n(q) = \lambda_n^+(q) - \lambda_n^-(q), \qquad n \geqslant 1,$$

of the potential q. If a gap length is zero, one speaks of a *collapsed gap*, otherwise of an *open gap*.

We recall that the gaps separate the *spectral bands*

$$B_n = \left[\lambda_{n-1}^+, \lambda_n^-\right], \qquad n \geqslant 1,$$

which are dynamically characterized as the locus of those real λ, for which all solutions of $Lf = \lambda f$ are bounded. Consequently, for any λ in the interior of an open gap as well as for all $\lambda < \lambda_0^+$, any nontrivial solution of $Lf = \lambda f$ is unbounded.

For complex q, the periodic eigenvalues are still well defined, but in general not real, since L is no longer self-adjoint. Their asymptotic behavior is the same, however, and we may order them lexicographically – first by their real part, then by their imaginary part – so that

$$\lambda_0(q) \prec \lambda_1^-(q) \preccurlyeq \lambda_1^+(q) \prec \cdots \prec \lambda_n^-(q) \preccurlyeq \lambda_n^+(q) \prec \cdots .$$

The gap lengths are then defined as before, but are now complex valued in general. They are also no longer characterized dynamically.

We are interested in the relationship between the regularity of a potential and the sequence of its gap lengths. Hochstadt [10] observed that

$$q \in C^\infty(S^1, \mathbb{R}) \quad \Leftrightarrow \quad \gamma_n(q) = O(n^{-k}) \text{ for all } k \geqslant 0,$$

and Marčenko & Ostrowskiĭ [13] subsequently showed that

$$q \in H^k(S^1, \mathbb{R}) \quad \Leftrightarrow \quad \sum_{n \geqslant 1} n^{2k} \gamma_n^2(q) < \infty$$

for all nonnegative integers k. Trubowitz [17] then proved that

$$q \in C^\omega(S^1, \mathbb{R}) \quad \Leftrightarrow \quad \gamma_n(q) = O(\mathrm{e}^{-an}) \text{ for some } a > 0.$$

Later, due to the realization of the periodic KdV flow as an isospectral deformation of Hill's operator, other regularity classes such as Gevrey functions and non-real potentials came into focus. Recent results in this direction are for example due to Sansuc & Tkachenko [15], Kappeler & Mityagin [11, 12] and Djakov & Mityagin [2, 3]. All this shows that within certain limits, one may think of the gap lengths as another kind of Fourier coefficients of the potential.

It is the purpose of this paper to further extend these results and to give a new, short, self-contained proof that applies simultaneously to all cases. This proof does not employ any conformal mappings, trace formula, asymptotic expansions, iterative arguments, or other convolutions. Instead, the essential ingredient is the inverse function theorem.

To set the stage, we introduce *weighted Sobolev spaces* $\mathcal{H}^w$ [11, 12]. A *normalized weight* is a function $w\colon \mathbb{Z} \to \mathbb{R}$ with

$$w_n = w_{-n}, \qquad w_n \geqslant 1$$

for all n, and the class of all such weights is denoted by $\mathcal{W}$. The w-norm $\|q\|_w$ of a complex 1-periodic function $q = \sum_{n\in\mathbb{Z}} q_n \mathrm{e}^{2n\pi i x}$ is then defined through

$$\|q\|_w^2 = \sum_{n\in\mathbb{Z}} w_n^2 \, |q_n|^2 ,$$

and

$$\mathcal{H}^w := \{q \in L^2(S^1, \mathbb{C}) : \|q\|_w < \infty\}$$

is the Banach space of all such functions with finite w-norm. Note that

$$\mathcal{H}^{\circ} := \bigcup_{w\in\mathcal{W}} \mathcal{H}^w = L^2(S^1, \mathbb{C}),$$

since all weights are assumed to be at least 1.

Here are some examples of relevant weights. Let $\langle n\rangle = |n| + 1$. For $r \geqslant 0$, the polynomial weights

$$w_n = \langle n\rangle^r$$

give rise to the usual Sobolev spaces $\mathcal{H}^r$. For $r \geqslant 0$ and $a \geqslant 0$, the exponential weights

$$w_n = \langle n\rangle^r \mathrm{e}^{a|n|}$$

give rise to spaces $\mathcal{H}^{r,a}$ of functions in L^2 analytic on the strip $|\mathrm{Im}\, z| < a/2\pi$ with traces in $\mathcal{H}^r$ on the boundary lines. In between are, among others, the subexponential weights

$$w_n = \langle n\rangle^r \mathrm{e}^{a|n|^\sigma}, \qquad 0 < \sigma < 1,$$

giving rise to Gevrey spaces $\mathcal{H}^{r,a,\sigma}$, and weights of the form

$$w_n = \langle n\rangle^r \exp\left(\frac{a\,|n|}{1 + \log^\alpha\langle n\rangle}\right), \qquad \alpha > 0.$$

More examples are given below.

For the most part we will be concerned with the subclass $\mathcal{M}$ of weights that are also *submultiplicative*. That is,

$$w_{n+m} \leqslant w_n w_m$$

for all n and m. This implies in particular that

$$\chi(w) := \lim_{n\to\infty} \frac{\log w_n}{n}$$

exists and is nonnegative [14, no. 98], so submultiplicative weights can not grow faster than exponentially. All the weights given above are submultiplicative, and

$$\mathcal{H}^{\omega} := \bigcap_{w\in\mathcal{M}} \mathcal{H}^{w}$$

is the space of all entire functions of period 1. It turns out that only in the submultiplicative case, and more precisely in the subexponential case, there is a one-to-one relationship between the decay rates of Fourier coefficients and spectral gap lengths.

We begin by considering the forward problem of controlling the gap lengths of a potential in terms of its regularity, first for submultiplicative weights – see [12]. We let

$$\mathcal{h}^{w} = \left\{ u = (u_n)_{n\geqslant 1} : \textstyle\sum_{n\geqslant 1} w_n^2 \left|u_n\right|^2 < \infty \right\}.$$

and $\gamma(q) = (\gamma_n(q))_{n\geqslant 1}$.

Theorem 1 *If $q \in \mathcal{H}^w$ with $w \in \mathcal{M}$, then $\gamma(q) \in \mathcal{h}^w$. In particular,*

$$\sum_{n\geqslant N} w_n^2 \left|\gamma_n(q)\right|^2 \leqslant 9 \left\|T_N q\right\|_w^2 + \frac{576}{N} \left\|q\right\|_w^4$$

for all $N \geqslant 4\left\|q\right\|_w$, where $T_N q = \sum_{|n|\geqslant N} q_n \mathrm{e}^{2n\pi i x}$.

We note in passing that finite-gap potentials are dense in $\mathcal{H}^w$ for $w \in \mathcal{M}$. More specifically, we call q an *N-gap potential*, if $\gamma_n(q) = 0$ for all $n > N$. We do not insist, however, that the first N gaps are all open.

Theorem 2 *The union of N-gap potentials is dense in $\mathcal{H}^w$ for $w \in \mathcal{M}$.*

We now turn to the converse problem of recovering the regularity of a potential from the asymptotic behavior of its gap lengths. Here the situation is not as clear cut as for the forward problem. Gasymov [6] observed that *any* L^2-potential of the form

$$q = \sum_{n\geqslant 1} q_n \mathrm{e}^{2n\pi i x} = \sum_{n\geqslant 1} q_n z^n \Big|_{z=\mathrm{e}^{2\pi i x}}$$

is a 0-gap potential. In the complex case, the gap sequence therefore need not contain *any* information about the regularity of the potential. But even in the real case the situation is not completely straightforward, as there are finite-gap potentials, that are not entire functions, but have poles. Thus, although in this case $\gamma_n \sim \mathrm{e}^{-an}$ for *all* $a > 0$, we have $q_n \sim \mathrm{e}^{-\alpha n}$ only for *some* $\alpha > 0$.

To obtain a true converse to Theorem 1 we need to exclude *exponential* weights, that is, submultiplicative weights w with $\chi(w) > 0$. We call a weight *subexponential*, if $\chi(w) = 0$ with

$$\frac{\log w(n)}{n} \to 0 \quad \text{as} \quad n \to \infty$$

in an eventually *monotone* manner, while $w(n)$ itself is assumed to be nondecreasing for $n \geqslant 0$. – The following theorem extends results of [2].

Theorem 3 *Suppose $q \in \mathcal{H}^0$ is real, and*

$$\gamma(q) \in \ell^w.$$

If w is subexponential, then $q \in \mathcal{H}^w$. If w is exponential, then q is real analytic.

This theorem does not extend to complex potentials because of Gasymov's observation. But Sansuc & Tkachenko [15] noted that the situation can be remedied by taking into account additional spectral data. In particular, they considered the quantities $\delta_n = \mu_n - \tau_n$, where μ_n denotes the Dirichlet eigenvalues of a potential and $\tau_n = (\lambda_n^+ + \lambda_n^-)/2$ the mid-points of its spectral gaps. Essentially, one needs that

- δ_n is continuously differentiable,
- δ_n vanishes whenever $\lambda_n^+ = \lambda_n^-$ has also geometric multiplicity 2, and
- there are real numbers ξ_n such that

$$\mathrm{d}\delta_n = t_n + O(1/n), \qquad t_n = \cos 2n\pi(x + \xi_n),$$

uniformly on bounded subsets of a complex neighbourhood of $\mathcal{H}^0_{\mathbb{R}}$ [1].

[1] That is, $\|\mathrm{d}_q\delta_n - t_n\|_0 \leqslant C_\delta(\|q\|_0)/n$ with C_δ depending only on $\|q\|_0 := \|q\|_{\mathcal{H}^0}$.

We then speak of *auxiliary gap lengths* on a neighbourhood of $\mathscr{H}^{\circ}_{\mathbb{R}}$.

For example, let σ_n denote the eigenvalues of the operator L with symmetric Sturm-Liouville boundary conditions

$$y \cos\alpha + y' \sin\alpha = 0 \quad \text{on} \quad \partial[0,1].$$

Dirichlet and Neumann boundary conditions correspond to the choices $\alpha = 0$ and $\alpha = \pi/2$, respectively. Then $\sigma_n \in [\lambda_n^-, \lambda_n^+]$ in the real case, and $\delta_n = \sigma_n - \tau_n$ are auxiliary gap lengths. – The following theorem extends results of [3, 15].

Theorem 4 *Let δ_n be a family of auxiliary gap lengths on $\mathscr{H}^{\circ}$.*
(i) *If $q \in \mathscr{H}^w$ with $w \in \mathcal{M}$, then $\delta(q) \in \hbar^w$. In particular,*

$$\sum_{n \geqslant N} w_n^2 \, |\delta_n(q)|^2 \leqslant 4 \, \|T_N q\|_w^2 + \frac{256}{N} \, \|q\|_w^4$$

for all N sufficiently large, where $T_N q = \sum_{|n| \geqslant N} q_n \mathrm{e}^{2n\pi i x}$.
(ii) *Conversely, suppose $q \in \mathscr{H}^{\circ}$ with*

$$\gamma(q) \in \hbar^w \quad \textit{and} \quad \delta(q) \in \hbar^w.$$

If w is subexponential, then $q \in \mathscr{H}^w$. If w is exponential, then q is analytic.

Following [3] we may consider $\lambda_n^-, \tau_n + \delta_n, \lambda_n^+$ as the vertices of a *spectral triangle* Δ_n, and

$$\Gamma_n(q) = |\gamma_n(q)| + |\delta_n(q)|$$

as a measure of its size, which takes the role of γ_n in the complex case. We then have the following consequence of Theorems 1 and 4.

Theorem 5 *If w is subexponential, then*

$$q \in \mathscr{H}^w \quad \Leftrightarrow \quad \Gamma(q) \in \hbar^w,$$

where Γ_n denotes the size of the n-th spectral triangle defined by the gap lengths γ_n and some auxiliary gap lengths δ_n.

We briefly look at the case of weights growing faster than exponentially, thus characterizing classes of entire functions. One can expect the gap lengths to decay faster than exponentially, too, albeit not at the same rate. We note a general result to this effect for *superexponential* weights, that is, weights w with $\chi(w) = \infty$. We only consider the gap lengths γ_n. The result for auxiliary gap lengths δ_n is exactly the same, only the lower bound for n has to be augmented. See also [4].

Theorem 6 *If $q \in \mathcal{H}^w$ with a superexponential weight $w \in \mathcal{W}$, then*

$$|\gamma_n(q)| \leqslant 2n \exp(-n\psi(\tilde{n})), \qquad \tilde{n} = \frac{n}{4\,\|q\|_w},$$

for all $n \geqslant 4\,\|q\|_w$, where $\psi(r) = \min_{m\geqslant 1} \dfrac{\log r w(m)}{m}$.

For instance, for $w_n = \exp(|n|^\sigma)$ with $\sigma > 1$ one has

$$\psi(\tilde{n}) = c_\sigma \log^{1-1/\sigma} \tilde{n}$$

with $c_\sigma = \sigma/(\sigma-1)^{1-1/\sigma}$. Djakov & Mityagin [4] construct an example showing that as far as the order in n is concerned, the resulting gap estimate can not be improved.

We point out that the preceding theorem is not optimal for trigonometric polynomials. Consider for example the Mathieu potential

$$q = \mu \cos 2\pi x, \qquad \mu > 0.$$

Using the just mentioned weight, we have $\|q\|_w = c\mu/4$ with a certain constant c for *all* $\sigma > 1$, and letting σ tend to infinity we obtain

$$\gamma_n(q) \leqslant 2n \exp\left(-n \log \frac{n}{c\mu}\right) = 2n \left(\frac{c\mu}{n}\right)^n.$$

But Harrell [9] and Avron & Simon [1] found the better exact asymptotics

$$\gamma_n(q) = 8\pi^2 \left(\frac{\mu}{8\pi^2}\right)^n \frac{1}{(n-1)!^2} \left(1 + O(n^{-2})\right),$$

This result was later extended by Grigis [8] to more general real trigonometric polynomials, and to their spectral triangles by Djakov & Mityagin [4]. These better estimates are obtained by directly evaluating an explicit representation of some coefficient – see the end of section 5. This approach is different from the one taking in this paper and will not be reproduced here.

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2 Outline

The idea of the proof of Theorem 1 is due to Kappeler & Mityagin [12]. They employ a Lyapunov-Schmidt reduction, called *Fourier block decomposition*.

The aim is to determine those λ near $n^2\pi^2$ with n sufficiently large, for which the equation $-y''+qy=\lambda y$ admits a nontrivial 2-periodic solution f. As q can be considered small for large n, one can expect its dominant modes to be $\mathrm{e}^{\pm n\pi ix}$. So it makes sense to separate these modes from the other ones by a Lyapunov-Schmidt reduction.

To this end we consider a Banach space $\mathcal{H}^w_\star$ of *2-periodic* functions, and write

$$\begin{aligned}\mathcal{H}^w_\star &= \mathcal{P}_n \oplus \mathcal{Q}_n \\ &= \operatorname{span}\{e_k : |k| = n\} \oplus \operatorname{span}\{e_k : |k| \neq n\},\end{aligned}$$

where $e_k = \mathrm{e}^{k\pi ix}$. The projections onto $\mathcal{P}_n$ and $\mathcal{Q}_n$ are denoted by P_n and Q_n, respectively. Then we write $-f''+qf=\lambda f$ in the form

$$A_\lambda f := f'' + \lambda f = Vf,$$

where V denotes the operator of multiplication with q. With

$$f = u + v = P_n f + Q_n f,$$

this equation decomposes into the two equations

$$\begin{aligned}A_\lambda u &= P_n V(u+v), \\ A_\lambda v &= Q_n V(u+v),\end{aligned}$$

called the P- and Q-equation, respectively, for some inscrutable reason.

We first solve the Q-equation by writing v as a function of u. This will reduce the P-equation to a two-dimensional equation with a 2×2 coefficient matrix S_n, which is singular precisely when λ is a periodic eigenvalue. The coefficients of S_n then provide all the data to prove Theorem 1, essentially as in [12].

To go beyond Theorem 1 – and this is the new ingredient – we regard these coefficients as analytic functions of their potential in $\mathcal{H}^0$, and employ them to define a near identity diffeomorphism Φ that introduces Fourier coefficients adapted to spectral gaps and preserves the regularity of potentials. That is, $p=\Phi(q)$ is in $\mathcal{H}^w$ if and only if q is in $\mathcal{H}^w$. This will be an immediate consequence of the inverse function theorem.

Establishing the regularity of a potential q then amounts to showing that $\Phi(q)$ is in $\mathcal{H}^w$. In the real case, this involves a geometric argument using the gap length asymptotics and a trick to temper the resulting w-norms. In the complex case, auxiliary gap lengths are needed in those cases where the coefficient matrix S_n is not close to a hermitian matrix to obtain the same conclusion.

3 Preparation

Given a weight w, we introduce the Banach space

$$\mathcal{H}_\star^w = \left\{ u = \textstyle\sum_{m\in\mathbb{Z}} u_m e_m : \|u\|_w < \infty \right\}$$

of complex functions of *period 2* and finite $\|\cdot\|_w$-norm,

$$\|u\|_w^2 = \sum_{m\in\mathbb{Z}} w_{m/2}^2 \, |u_m|^2 .$$

We assume for simplicity, and without noticeable loss of generality, that the weights are also defined on $\mathbb{Z}/2$ and have the same properties there. Obviously, $\mathcal{H}_\star^w$ is an extension of $\mathcal{H}^w$. On $\mathcal{H}_\star^w$ we consider operator norms that are defined in terms of *shifted w-norms*

$$\|u\|_{w;i}^2 = \|u e_i\|_w^2 = \sum_{m\in\mathbb{Z}} w_{m/2}^2 \, |u_{m-i}|^2 .$$

Finally, we introduce the disjoint vertical complex strips

$$U_n = \left\{ \lambda \in \mathbb{C} : \left|\operatorname{Re}\lambda - n^2\pi^2\right| \leqslant 12n \right\}, \qquad n \geqslant 1.$$

Recall that q is identified with the operator V of multiplication by q.

Lemma 1 *If $q \in \mathcal{H}^w$ with $w \in \mathcal{M}$, then for $n \geqslant 1$ and $\lambda \in U_n$,*

$$T_n = V A_\lambda^{-1} Q_n$$

is a bounded linear operator on $\mathcal{H}_\star^w$ with norm

$$\|T_n\|_{w;i} \leqslant \frac{2}{n} \|q\|_w , \qquad i \in \mathbb{Z}.$$

Proof. We have $A_\lambda e_m = (\lambda - m^2\pi^2) e_m$ for all m, and for $|m| \neq n$, one checks that

$$\min_{\lambda\in U_n} \left|\lambda - m^2\pi^2\right| \geqslant \left|n^2 - m^2\right| > 0.$$

Therefore, the restriction of A_λ to the range of Q_n is boundedly invertible for all $\lambda \in U_n$, and for $f = \sum_{m\in\mathbb{Z}} f_m e_m$,

$$g = A_\lambda^{-1} Q_n f = \sum_{|m|\neq n} \frac{f_m}{\lambda - m^2\pi^2}\, e_m$$

is well defined. For the weighted L^1-norm $\|g\|_{w,1} = \sum_{m\in\mathbb{Z}} w_{m/2}\,|g_m|$ of g we then obtain, with the help of Hölder's inequality and the preceding two lines,

$$\begin{aligned} \|ge_i\|_{w,1} &\leqslant \sum_{|m|\neq n} \frac{w_{(m+i)/2}\,|f_m|}{|n^2 - m^2|} \\ &\leqslant \|f\|_{w;i} \left(\sum_{|m|\neq n} \frac{1}{|m^2 - n^2|^2} \right)^{1/2}. \end{aligned}$$

With

$$\sum_{|m|\neq n} \frac{1}{|m^2 - n^2|^2} \leqslant \frac{2}{n^2} \sum_{m\geqslant 1} \frac{1}{m^2} \leqslant \frac{4}{n^2},$$

we thus have $\|ge_i\|_{w,1} \leqslant 2n^{-1}\,\|f\|_{w;i}$. Finally, with $q = \sum_{m\in\mathbb{Z}} u_m e_m$,

$$(Vg)e_i = \sum_{m\in\mathbb{Z}} e_{m+i} \sum_{l\in\mathbb{Z}} u_{m-l}\, g_l = \sum_{m\in\mathbb{Z}} e_m \sum_{l\in\mathbb{Z}} u_{m-l}\, g_{l-i} = V(ge_i)$$

and thus $(T_n f)e_i = (Vg)e_i = V(ge_i)$. Standard estimates for the convolution of two sequences and the submultiplicity of the weights then give

$$\|T_n f\|_{w;i} = \|V(ge_i)\|_w \leqslant \|V\|_w\,\|ge_i\|_{w,1} \leqslant \frac{2}{n}\,\|q\|_w\,\|f\|_{w;i}\,.$$

This holds for any $f \in \mathcal{H}_\star^w$ and any $i \in \mathbb{Z}$, so the claim follows. ∎

Thus, if $n \geqslant 4\,\|q\|_w$ and $w \in \mathcal{M}$, then T_n is a $\frac{1}{2}$-contraction on $\mathcal{H}_\star^w$ in particular with respect to the shifted norms $\|\cdot\|_{w;\pm n}$. It is this property that we actually need in section 5 to bound the n-th gap lengths from above.

4 Reduction

Multiplying the Q-equation from the left with VA_λ^{-1} we obtain

$$Vv = T_n Vu + T_n Vv, \qquad T_n = VA_\lambda^{-1} Q_n.$$

If T_n is a contraction on $\mathcal{H}_\star^w$, then this equation has a unique solution, namely

$$Vv = \hat{T}_n T_n Vu, \qquad \hat{T}_n = (I - T_n)^{-1}.$$

Inserted into the P-equation this gives

$$A_\lambda u = P_n Vu + P_n \hat{T}_n T_n Vu = P_n \hat{T}_n Vu.$$

So the P- and Q-equation reduce to S-equation

$$S_n u = 0, \qquad S_n = A_\lambda - P_n \hat{T}_n V.$$

Any nontrivial solution u gives rise to a 2-periodic solution of $A_\lambda f = Vf$, and vice versa. Hence, a complex number λ near $n^2\pi^2$ is a periodic eigenvalue of q if and only if the determinant of S_n vanishes.

The matrix representation of any operator I on the two-dimensional space $\mathcal{P}_n$ is given by $(\langle Ie_{\pm n}, e_{\pm n}\rangle)$, where $\langle f, g\rangle = \int_0^1 f\bar{g}\,\mathrm{d}x$. We find that

$$A_\lambda = \begin{pmatrix} \lambda - \sigma_n & 0 \\ 0 & \lambda - \sigma_n \end{pmatrix}, \qquad P_n \hat{T}_n V = \begin{pmatrix} a_n & c_n \\ c_{-n} & a_{-n} \end{pmatrix},$$

with $\sigma_n = n^2\pi^2$ and

$$a_n = \langle \hat{T}_n Ve_n, e_n\rangle, \qquad c_n = \langle \hat{T}_n Ve_{-n}, e_n\rangle.$$

Moreover, looking at the series expansion of $\hat{T}_n$ one checks that $(\hat{T}_n V)^* = (\hat{T}_n V)^-$, the complex conjugate of $\hat{T}_n V$. Therefore,

$$\begin{aligned} a_n &= \langle \hat{T}_n Ve_n, e_n\rangle \\ &= \langle e_n, (\hat{T}_n V)^- e_n\rangle \\ &= \langle e_n, (\hat{T}_n Ve_{-n})^-\rangle \\ &= \langle \hat{T}_n Ve_{-n}, e_{-n}\rangle = a_{-n}. \end{aligned}$$

That is, the diagonal of S_n is homogeneous, and we have

$$S_n = \begin{pmatrix} \lambda - \sigma_n - a_n & -c_n \\ -c_{-n} & \lambda - \sigma_n - a_n \end{pmatrix}.$$

Incidentally, at this point we may recover Gasymov's observation for complex potentials of the form

$$q = \sum_{m\geqslant 1} q_m e_{2m} = \sum_{m\geqslant 1} q_m \mathrm{e}^{2m\pi i x}$$

In that case, $\hat{T}_n V e_n$ is given by a power series in $\mathrm{e}^{2\pi i x}$ with lowest term e_{n+2}, whence $a_n = c_n = 0$ and

$$S_n = \begin{pmatrix} \lambda - \sigma_n & 0 \\ -c_{-n} & \lambda - \sigma_n \end{pmatrix}.$$

It follows that $\lambda_n^{\pm} = \sigma_n$ for all $n \geqslant 1$, which is the claim.

5 Gap Estimates

Lemma 2 *If $n \geqslant 4\,\|q\|_w$, then*

$$|a_n - q_0|_{U_n}, \; w_n\,|c_n - q_n|_{U_n}, \; w_n\,|c_{-n} - q_{-n}|_{U_n} \leqslant 2\,\|T_n\|_{w;n}\,\|q\|_w\,.$$

Proof. Consider $c_n = \langle \hat{T}_n V e_{-n}, e_n \rangle$. We note that $\hat{T}_n = I + \hat{T}_n T_n$ and thus $c_n = q_n + \langle \hat{T}_n T_n V e_{-n}, e_n \rangle$. In general, from $\langle f, e_n \rangle = \langle f e_n, e_{2n} \rangle$ we obtain

$$w_n\,|\langle f, e_n \rangle| \leqslant \|f e_n\|_w = \|f\|_{w;n}\,.$$

The claim then follows with $f = \hat{T}_n T_n V e_{-n}$,

$$\|\hat{T}_n T_n V e_{-n}\|_{w;n} \leqslant 2\,\|T_n\|_{w;n}\,\|V e_{-n}\|_{w;n}\,,$$

and $\|V e_{-n}\|_{w;n} = \|V e_0\|_w = \|q\|_w$. The rest is proven analogously. ∎

Remark. We have to make this somewhat roundabout argument, since composition and multiplication are not associative. That is, $(T_n V e_n) e_m$ is *not equal* to $T_n V e_{n+m}$, and therefore $\langle \hat{T}_n V e_{-n}, e_n \rangle$ is *not equal* to $\langle \hat{T}_n q, e_{2n} \rangle$. The estimate of the latter would be much more straightforward and would not require shifted norms.

Lemma 3 *If $n \geqslant 4\,\|q\|_w$, then the determinant of S_n has exactly two complex roots ξ_-, ξ_+ in U_n, which are contained in*

$$D_n = \{\lambda : |\lambda - \sigma_n| \leqslant 6\,\|q\|_w\}$$

and satisfy

$$|\xi_+ - \xi_-|^2 \leqslant 9\,|c_n c_{-n}|_{U_n}.$$

A more precise location of these roots is obtained in the proof of Lemma 10 below. But for now, this simpler statement suffices.

Proof. Write $\det S_n = g_+ g_-$ with

$$g_\pm = \lambda - \sigma_n - a_n \mp \varphi_n, \qquad \varphi_n = \sqrt{c_n c_{-n}},$$

where the choice of the branch of the root is immaterial. With $n \geqslant 4\,\|q\|_w$ we can apply Lemma 2 and obtain

$$|a_n - q_0|_{U_n},\, w_n\,|c_n - q_n|_{U_n} \leqslant \|q\|_w.$$

Since $w_n \geqslant 1$ we conclude that

$$|a_n|_{U_n} + |\varphi_n|_{U_n} \leqslant 4\,\|q\|_w < 2n \leqslant \inf_{U_n \setminus D_n} |\lambda - \sigma_n|.$$

It follows with topological degree theory that both g_+ and g_- have exactly one root in D_n, while they obviously have no roots in $U_n \setminus D_n$.

To estimate the distance of these roots, let ξ_+ be the root of g_+, and

$$K_n = \{\lambda : |\lambda - \xi_+| \leqslant 3r_n\}, \qquad r_n := |\varphi_n|_{U_n} \leqslant 2\,\|q\|_w \leqslant n.$$

The function $h = \lambda - \sigma_n - a_n(\xi_+) - \varphi_n(\xi_+)$ vanishes at ξ_+, thus $|h|\Big|_{\partial K_n} = 3r_n$. On the other hand,

$$|h - g_-|_{\partial K_n} \leqslant |a_n - a_n(\xi_+)|_{K_n} + 2\,|\varphi_n|_{U_n} < r_n + 2r_n = |h|\,\Big|_{\partial K_n},$$

since $|\partial_\lambda a_n|_{K_n} \leqslant |a_n|_{U_n}/4n \leqslant 1/4$ by Cauchy's inequality. It follows again with topological degree theory that g_- has on K_n the same index with respect to 0 as h, namely 1. Hence, the second root ξ_- of $\det S_n$ is located in K_n, which gives the claim. ∎

We now prove Theorem 1. If $q \in \mathcal{H}^w$ with $w \in \mathcal{M}$ and $n \geqslant 4\,\|q\|_w$, then Lemma 3 applies, giving us two roots of $\det S_n$ in $D_n \subset U_n$. Now the union of all strips U_n covers the right complex half plane. Since $\lambda_n^\pm \sim n^2\pi^2$ asymptotically, and since there are no periodic eigenvalues in $\bigcup_{n\geqslant 4\|q\|_w}(U_n \setminus D_n)$, those two roots in D_n must be the periodic eigenvalues $\lambda_n^\pm$. Thus,

$$|\gamma_n(q)|^2 = |\xi_+ - \xi_-|^2 \leqslant 9\,|c_n c_{-n}|_{U_n} \leqslant \frac{9}{2}\,|c_n|_{U_n}^2 + \frac{9}{2}\,|c_{-n}|_{U_n}^2 .$$

By Lemmas 1 and 2,

$$w_n\,|c_n| \leqslant w_n\,|q_n| + 2\,\|T_n\|_{w;-n}\,\|q\|_w \leqslant w_n\,|q_n| + \frac{4}{n}\,\|q\|_w^2 .$$

Both estimates together then lead to

$$\frac{1}{9}w_n^2\,|\gamma_n(q)|^2 \leqslant w_n^2\,|q_n|^2 + w_n^2\,|q_{-n}|^2 + \frac{32}{n^2}\,\|q\|_w^4 .$$

Summing up, we arrive at

$$\begin{aligned} \frac{1}{9}\sum_{n\geqslant N} w_n^2\,|\gamma_n(q)|^2 &\leqslant \sum_{|n|\geqslant N} w_n^2\,|q_n|^2 + \|q\|_w^4 \sum_{n\geqslant N}\frac{32}{n^2} \\ &\leqslant \|T_N q\|_w^2 + \frac{64}{N}\,\|q\|_w^4 . \end{aligned}$$

This gives Theorem 1.

Incidentally, if we just make use of $w_n\,|c_n| \leqslant 2\,\|q\|_w$, then we get the individual gap estimate

$$w_n\,|\gamma_n(q)| \leqslant 6\,\|q\|_w .$$

We will use this observation in section 10. Finally, we note that Lemma 3 together with the expansion

$$c_n = \langle \hat{T}_n V e_{-n}, e_n\rangle = \sum_{\nu\geqslant 0}\langle T_n^\nu V e_{-n}, e_n\rangle$$

allows for an effective control of γ_n for trigonometric polynomials q, since then some first terms in the series vanish – see [1, 4].

6 Adapted Fourier Coefficients

The 2×2-matrix S_n contains all the information we need about the n-th periodic eigenvalues of a potential, at least in the real case. Even more to the point, the diagonal of S_n vanishes at a unique point

$$\lambda = \alpha_n(q),$$

and it suffices to consider its off-diagonal elements at this value of λ. We will make use of these values to define a real analytic *adapted Fourier coefficient map*, which allows us to prove the regularity results by invoking the inverse function theorem.

We begin by observing that the coefficients a_n and c_n do not depend on the underlying space $\mathcal{H}^w$, but are rather defined on appropriate balls in $\mathcal{H}^{\mathrm{o}}$, with estimates depending on the regularity of q. To make this precise, we introduce the notation

$$B_m^w = \left\{ q \in \mathcal{H}^w : 4 \left\| q \right\|_w \leqslant m \right\}$$

and note that $B_m^w \subset B_m^{\mathrm{o}} := \{ q \in \mathcal{H}^{\mathrm{o}} : 4 \left\| q \right\|_{\mathrm{o}} \leqslant m \} \subset \mathcal{H}^{\mathrm{o}}$ for all $w \in \mathcal{M}$.

We assume from now on that $[q] = \int_0^1 q(x)\,\mathrm{d}x = q_0 = 0$, since adding a constant to the potential q shifts its entire spectrum by this amount, but does not affect the lengths of its gaps.

Lemma 4 *For $n \geqslant m$, the coefficients a_n and c_n are analytic functions on $U_n \times B_m^{\mathrm{o}}$ with*

$$|a_n|_{U_n \times B_m^{\mathrm{o}}}, \; w_n \, |c_n - q_n|_{U_n \times B_m^w}, \; w_n \, |c_{-n} - q_{-n}|_{U_n \times B_m^w} \leqslant \frac{m^2}{4n}$$

for all weights $w \in \mathcal{M}$.

Proof. The estimates follow from Lemmas 1 and 2, the normalization $q_0 = 0$ and $\left\| q \right\|_w \leqslant m/4$ on B_m^w. The analytic dependence on q then follows from the series expansion of $\hat{T}_n$. ∎

Lemma 5 *For $m \geqslant 1$ and each $n \geqslant m$, there exists a unique real analytic function*

$$\alpha_n : \; B_m^{\mathrm{o}} \to \mathbb{C}, \qquad |\alpha_n - \sigma_n|_{B_m^{\mathrm{o}}} \leqslant \frac{m^2}{4n},$$

such that $\lambda - \sigma_n - a_n(\lambda, \cdot)|_{\lambda = \alpha_n} \equiv 0$ identically on B_m^{o}.

Proof. Consider the fixed point problem for the operator T,

$$T\alpha := \sigma_n + a_n(\alpha, \cdot),$$

on the ball of all real analytic functions $\alpha\colon B_m^\circ \to \mathbb{C}$ with $|\alpha - \sigma_n|_{B_m^\circ} \leqslant m^2/4n$. Since clearly $m^2/4n \leqslant n$ by assumption, each such function α maps B_m° into the disc $D_n = \{|\lambda - \sigma_n| \leqslant n\} \subset U_n$, and so

$$|T\alpha - \sigma_n|_{B_m^\circ} \leqslant |a_n|_{U_n} \leqslant m^2/4n$$

in view of Lemma 4. Moreover, T contracts by a factor

$$|\partial_\lambda a_n|_{D_n} \leqslant \frac{|a_n|_{U_n}}{2n} \leqslant \frac{m^2}{8n^2} \leqslant \frac{1}{8},$$

using Cauchy's estimate. Hence, we find a unique fixed point $\alpha_n = T\alpha_n$ with the properties as claimed. ∎

In the following we let $\alpha_{-n} = \alpha_n$ to simplify notation. – For each $m \geqslant 1$ we now define a map Φ_m on B_m° by

$$\Phi_m(q) = \sum_{|n|<M_m} q_n e_{2n} + \sum_{|n|\geqslant M_m} c_n(\alpha_n(q), q) e_{2n},$$

where $M_m = 2^{10}m^2$. Thus, for $|n| \geqslant M_m$ the Fourier coefficients of the 1-periodic function $p = \Phi_m(q)$ are $p_n = c_n(\alpha_n)$, and

$$S_n(\alpha_n, q) = \begin{pmatrix} 0 & -p_n \\ -p_{-n} & 0 \end{pmatrix}.$$

These new Fourier coefficients are adapted to the lengths of the corresponding spectral gaps, whence we call Φ_m the *adapted Fourier coefficient map* on B_m°.

Proposition 6 *For each $m \geqslant 1$, Φ_m maps B_m° into $\mathcal{H}^\circ$, and for every weight $w \in \mathcal{M}$, its restrictions to B_m^w is a real analytic diffeomorphism*

$$\Phi_m\big|\, B_m^w:\ \ B_m^w \to \Phi_m(B_m^w) \subset \mathcal{H}^w,$$

such that

$$\frac{1}{2}\,\|q\|_w \leqslant \|\Phi_m(q)\|_w \leqslant 2\,\|q\|_w\,, \qquad q \in B_m^w.$$

Moreover, $\|D\Phi_m - I\|_{B_m^w} \leqslant 1/8$.

Proof. Since α_n maps B°_{2m} into U_n for $n \geqslant 2m$, each coefficient $c_n(\alpha_n(q), q)$ is well defined for $q \in B^{\circ}_{2m}$, and

$$w_n \, |c_n(\alpha_n) - q_n|_{B^w_{2m}} \leqslant w_n \, |c_n - q_n|_{U_n \times B^w_{2m}} \leqslant \frac{m^2}{n}$$

by Lemma 4. Hence the map Φ_m is defined on B°_{2m}, and

$$\begin{aligned} \|\Phi_m - \mathrm{id}\|^2_{w,B^w_{2m}} &= \sum_{|n| \geqslant M_m} w_n^2 \, |c_n(\alpha_n) - q_n|^2_{B^w_{2m}} \\ &\leqslant \sum_{n \geqslant M_m} \frac{2m^4}{n^2} \leqslant \frac{4m^4}{M_m} = \frac{m^2}{256} \end{aligned}$$

by our choice of M_m. Therefore, $\Phi_m : \, B^w_{2m} \to \mathcal{H}^w$ with $\|\Phi_m - \mathrm{id}\|_{w,B^w_{2m}} \leqslant \dfrac{m}{16}$. Cauchy's estimate then yields

$$\|D\Phi_m - I\|_{w,B^w_m} \leqslant \frac{2}{m} \, \|\Phi_m - \mathrm{id}\|_{w,B^w_{2m}} \leqslant \frac{1}{8}.$$

Now the result follows by standard arguments and the fact that $\Phi_m(0) = 0$. ▮

We now proof Theorem 2. Fix any ball $B_m = B^w_m$. The n-th gap of $q \in B_m$, with $n \geqslant M_m$, is collapsed if the n-th and $-n$-th Fourier coefficients of $\Phi_m(q)$ vanish, since then S_n vanishes identically at $\lambda = \alpha_n(q)$. Consequently, if

$$\Phi_m(q) \in \mathcal{G}_N = \mathrm{span}\,\{e_{2k} : |k| \leqslant N\},$$

N sufficiently large, then $q \in B_m$ is an N-gap potential. The union of the spaces $\mathcal{G}_N$ is dense in $\mathcal{H}^w$. Since Φ_m is a diffeomorphism on B_m, the family of N-gap potentials in B_m is also dense. Since B_m was arbitrary, this proves the theorem.

7 Regularity: The Abstract Case

From an abstract point of view, establishing the regularity of a potential q amounts to the following observation about its adapted Fourier coefficients.

Proposition 7 *If $q \in B^{\circ}_m$ for some $m \geqslant 1$, and*

$$\Phi_m(q) \in B^w_{m/2}$$

for some weight $w \in \mathcal{M}$, then $q \in B^w_m \subset \mathcal{H}^w$.

Proof. The map Φ_m is defined on B_m° and a real analytic diffeomorphism onto its image

$$\tilde{B}_m^{\circ} = \Phi_m(B_m^{\circ}) \subset \mathcal{H}^{\circ}.$$

At the same time, for any weight $w \in \mathcal{M}$, Φ_m is also defined on $B_m^w \subset B_m^{\circ} \cap \mathcal{H}^w$ and a real analytic diffeomorphism onto its image

$$\tilde{B}_m^w = \Phi_m(B_m^w) \subset \mathcal{H}^w.$$

Moreover, this image contains $B_{m/2}^w$ by Proposition 6. Thus, if Φ_m maps $q \in B_m^{\circ}$ to

$$p = \Phi_m(q) \in B_{m/2}^w,$$

then we must have

$$q = \Phi_m^{-1}\big|\, B_{m/2}^w(p) \in B_m^w \subset \mathcal{H}^w,$$

thus establishing the regularity of q. ∎

8 Regularity: The Real Case

Proposition 8 *Suppose $q \in B_m^{\circ}$ for some $m \geqslant 1$, and*

$$\Phi_m(q) \in \mathcal{H}^w.$$

If w is subexponential, then also $q \in \mathcal{H}^w$. If w is exponential, then $q \in \mathcal{H}^{v_\varepsilon}$ for all sufficiently small positive ε, where $v_\varepsilon = \mathrm{e}^{\varepsilon|\cdot|}$.

Note that in contrast to Proposition 7 we do *not* assume that $\Phi_m(q) \in B_{m/2}^w$. That is, we have no *a priori* bound on $\|\Phi_m(q)\|_w$. To reduce the situation to the former setting nonetheless, we introduce a modified weight w_ε, which tempers a potentially large chunk of $\|\Phi_m(q)\|_w$ arising from finitely many modes, without affecting the asymptotic behavior of w in the case of subexponential weights. The crucial ingredient is the following lemma.

Lemma 9 *If w is either subexponential or exponential, then*

$$w_\varepsilon := \min(v_\varepsilon, w) \in \mathcal{M}$$

for all sufficiently small positive ε.

Proof. If w is exponential, then $w_\varepsilon = v_\varepsilon$ for all sufficiently small positive ε, and there is nothing to do.

So assume w is subexponential. All the required properties are readily verified for w_ε, except submultiplicity. To do this, let

$$\tilde{w} = \log w, \qquad \tilde{w}_\varepsilon = \log w_\varepsilon.$$

As $\tilde{w}(n)/n$ converges eventually monotonically to zero by assumption, there exists for each sufficiently small $\varepsilon > 0$ an integer N_ε such that

$$\frac{\tilde{w}(i)}{i} \geqslant \varepsilon > \frac{\tilde{w}(n)}{n} > \frac{\tilde{w}(m)}{m} \quad \text{for} \quad 1 \leqslant i \leqslant N_\varepsilon < n < m.$$

It follows that

$$\tilde{w}_\varepsilon = \begin{cases} \tilde{v}_\varepsilon & \text{on } [0, N_\varepsilon], \\ \tilde{w} & \text{on } (N_\varepsilon, \infty). \end{cases}$$

To check for the subadditivity of $\tilde{w}_\varepsilon$ for $0 \leqslant n \leqslant m$, we consider the four possible cases

(a) $n + m \leqslant N_\varepsilon$, (c) $n \leqslant N_\varepsilon < m$,

(b) $m \leqslant N_\varepsilon < n + m$, (d) $N_\varepsilon < n$.

Case (a) reduces to $\tilde{v}_\varepsilon$, and case (d) reduces to $\tilde{w}$. In case (b),

$$\tilde{w}_\varepsilon(n+m) \leqslant \tilde{v}_\varepsilon(n+m) = \tilde{v}_\varepsilon(n) + \tilde{v}_\varepsilon(m) = \tilde{w}_\varepsilon(n) + \tilde{w}_\varepsilon(m).$$

Finally, in case (c), using the monotonicity property in the second line,

$$\begin{aligned} \tilde{w}_\varepsilon(n+m) &= \frac{\tilde{w}(n+m)}{n+m} n + \frac{\tilde{w}(n+m)}{n+m} m \\ &\leqslant \varepsilon n + \tilde{w}(m) \\ &= \tilde{w}_\varepsilon(n) + \tilde{w}_\varepsilon(m). \end{aligned}$$

This establishes the subadditivity of $\tilde{w}_\varepsilon$ for nonnegative arguments. The remaining cases all reduce to the monotonicity of $\tilde{w}_\varepsilon$, that is,

$$\tilde{w}_\varepsilon(n-m) \leqslant \tilde{w}_\varepsilon(n+m) \leqslant \tilde{w}_\varepsilon(n) + \tilde{w}_\varepsilon(m)$$

for $0 \leqslant m \leqslant n$. ∎

Proof of Proposition 8. We may assume that $q \neq 0$, and $m \geqslant 32 \|q\|_\circ$, since the assumptions are not affected by increasing m. For $p = \Phi_m(q)$ we have

$$\|p\|_\circ \leqslant 2 \|q\|_\circ$$

by Proposition 6. On the other hand, $p \in \mathcal{H}^w$ by assumption, so $\|p\|_w < \infty$. We can therefore choose N so large that

$$\|T_N p\|_w \leqslant \|p\|_\circ ,$$

where $T_N p = \sum_{|n| \geqslant N} p_n e_{2n}$. With respect to the weight $w_\varepsilon = \min(v_\varepsilon, w)$ with $\varepsilon \leqslant 1/2N$ sufficiently small, we then have

$$\begin{aligned} \|p\|_{w_\varepsilon}^2 &= \|p - T_N p\|_{w_\varepsilon}^2 + \|T_N p\|_{w_\varepsilon}^2 \\ &\leqslant \|p - T_N p\|_{v_\varepsilon}^2 + \|T_N p\|_w^2 \\ &\leqslant \mathrm{e}^{2N\varepsilon} \|p\|_\circ^2 + \|p\|_\circ^2 \\ &\leqslant 4 \|p\|_\circ^2 , \end{aligned}$$

or

$$4 \|p\|_{w_\varepsilon} \leqslant 8 \|p\|_\circ \leqslant 16 \|q\|_\circ \leqslant \frac{m}{2}.$$

Thus, $p \in B_{m/2}^{w_\varepsilon}$, whence

$$q = \Phi_m^{-1}(p) \in B_m^{w_\varepsilon} \subset \mathcal{H}^{w_\varepsilon}$$

by Proposition 7. The claim follows by noting that $\mathcal{H}^{w_\varepsilon} = \mathcal{H}^w$ for subexponential weights, and $\mathcal{H}^{w_\varepsilon} \subset \mathcal{H}^{v_\varepsilon}$ for exponential weights and all small $\varepsilon > 0$. ∎

To obtain Theorem 3 from Proposition 8, we now want to bound the Fourier coefficients of $p = \Phi_m(q)$ in terms of the gap lengths of q. For real q, this is fairly straightforward, since then

$$S_n = \begin{pmatrix} \lambda - \sigma_n - a_n & -c_n \\ -c_{-n} & \lambda - \sigma_n - a_n \end{pmatrix}$$

is hermitian, and $\det S_n$ is a real function of λ, which is close to the standard parabola with minimum near α_n and minimal value about $-p_n p_{-n} = -|p_n|^2$. The distance of its two roots is then about $|p_n|$. With foresight to the complex case, however, we want to consider a more general situation.

Lemma 10 *Let $q \in B_m^{\circ}$ for some $m \geqslant 1$ and $p = \Phi_m(q)$. If*

$$\frac{1}{4} \leqslant \left| \frac{p_n}{p_{-n}} \right| \leqslant 4$$

for any $n \geqslant M_m$, then

$$|p_n p_{-n}| \leqslant |\gamma_n(q)|^2 \leqslant 9\,|p_n p_{-n}|\,.$$

Proof. As in the proof of Lemma 3, write $\det S_n = g_+ g_-$ with

$$g_\pm = \lambda - \sigma_n - a_n \mp \varphi_n, \qquad \varphi_n = \sqrt{c_n c_{-n}}.$$

The assumptions imply that

$$\xi_n := \varphi_n(\alpha_n) = \sqrt{p_n p_{-n}} \neq 0, \qquad r_n := |\xi_n| > 0,$$

so we may choose a fixed sign of the root locally around α_n.

We compare g_+ with $h_+ = \lambda - \sigma_n - a_n(\alpha_n) - \varphi_n(\alpha_n)$ on the disc

$$D_n^+ = \{\lambda : |\lambda - (\alpha_n + \xi_n)| \leqslant r_n/2\}\,.$$

As $h_+(\alpha_n + \xi_n) = \xi_n - \varphi_n(\alpha_n) = 0$, we have

$$|h_+|\Big|_{\partial D_n^+} = \frac{r_n}{2}.$$

On the other hand, we momentarily show that on $D_n^{\circ} = \{\lambda : |\lambda - \alpha_n| \leqslant 2r_n\}$,

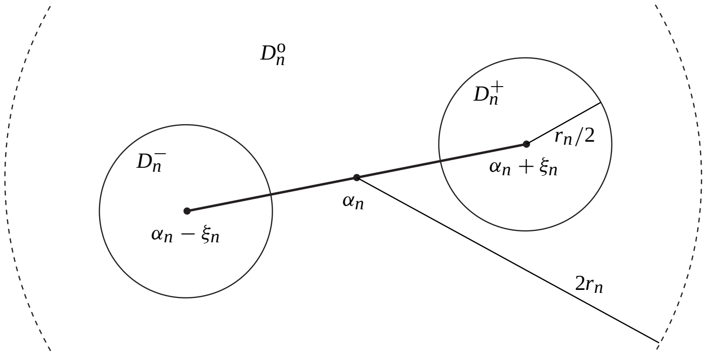
$$|\partial_\lambda a_n|_{D_n^{\circ}} \leqslant \frac{1}{18}, \qquad |\partial_\lambda \varphi_n|_{D_n^{\circ}} \leqslant \frac{1}{6},$$

which will give

$$\begin{aligned} |h_+ - g_+|_{D_n^+} &\leqslant |a_n - a_n(\alpha_n)|_{D_n^{\circ}} + |\varphi_n - \varphi_n(\alpha_n)|_{D_n^{\circ}} \\ &\leqslant \frac{r_n}{9} + \frac{r_n}{3} \\ &< \frac{r_n}{2} = |h_+|\Big|_{\partial D_n^+}\,. \end{aligned}$$

It follows that the unique root of g_+ within D_n must be contained in D_n^+, that is,

$$\xi_+ = \lambda_n^+ \in D_n^+.$$



Similarly, $\xi_- = \lambda_n^- \in D_n^- = \{\lambda : |\lambda - (\alpha_n - \xi_n)| \leqslant r_n/2\}$. Since $|\xi_n| = r_n$, we conclude that

$$r_n \leqslant |\gamma_n| = \left|\lambda_n^+ - \lambda_n^-\right| \leqslant 3r_n,$$

which is the claim.

It remains to prove the estimates for $\partial_\lambda a_n$ and $\partial_\lambda \varphi_n$. In view of Lemma 4 and Cauchy's inequality,

$$|\partial_\lambda a_n|_{D_n^\circ}, |\partial_\lambda c_n|_{D_n^\circ} \leqslant \frac{1}{36},$$

since the distance of D_n° to the boundary of U_n is at least $9n$. With $c_n(\alpha_n) = p_n$, $r_n = \sqrt{|p_n p_{-n}|}$ and the hypotheses of the lemma, we get

$$|c_n - p_n|_{D_n^\circ} \leqslant \frac{r_n}{16},$$

or

$$\frac{r_n}{2} \leqslant |p_n| \leqslant 2r_n.$$

Hence,

$$\frac{7}{16} r_n = \left(\frac{1}{2} - \frac{1}{16}\right) r_n \leqslant |c_n|\Big|_{D_n^\circ} \leqslant \left(2 + \frac{1}{16}\right) r_n = \frac{33}{16} r_n,$$

and therefore

$$\left|\frac{c_n}{c_{-n}}\right|_{D_n^\circ}, \left|\frac{c_{-n}}{c_n}\right|_{D_n^\circ} \leqslant 6.$$

Differentiating $\varphi_n = \sqrt{c_n c_{-n}}$ with respect to λ we finally obtain

$$|\partial_\lambda \varphi_n|_{D_n^\circ} \leqslant 3\left(|\partial_\lambda c_n|_{D_n^\circ} + |\partial_\lambda c_{-n}|_{D_n^\circ}\right) \leqslant \frac{1}{6}$$

as claimed. This completes the proof. ∎

We now prove Theorem 3. Suppose $q \in \mathcal{H}^\circ$ is real, and its gap lengths satisfy

$$\sum_{n\geqslant 1} w_n^2 \, |\gamma_n(q)|^2 < \infty.$$

Fix $m \geqslant 4\,\|q\|_\circ$, and consider the coefficients $p_n = c_n(\alpha_n)$ for $|n| \geqslant M_m$. As q is real, $p_{-n} = \bar{p}_n$. So the preceding lemma applies, giving

$$|p_{-n}| = |p_n| \leqslant |\gamma_n(q)|, \qquad n \geqslant M_m.$$

But this means that $p = \Phi_m(q) \in \mathcal{H}^w$, and the result follows with Proposition 8.

9 Regularity: The Complex Case

Let δ_n be a family of auxiliary gap lengths. Since the involved constant C_δ is supposed to depend only on $\|q\|_\circ$, we have on any ball B_m° an estimate

$$\left\|\mathrm{d}_q\delta_n - t_n\right\|_\circ \leqslant \frac{C_m}{n}, \qquad t_n = \cos 2n\pi(x+\xi_n).$$

with a constant C_m depending only on m.

Lemma 11 *If $q \in B_m^\circ$ for some $m \geqslant 1$ and $p = \Phi_m(q)$, then*

$$|\delta_n(q) - (\kappa p_n + \bar{\kappa} p_{-n})| \leqslant \frac{1}{4}(|p_n| + |p_{-n}|)$$

for $n \geqslant N_m := \max(M_m, 16\,C_m)$, where $\kappa = \mathrm{e}^{2n\pi i\xi_n}/2$.

Proof. Given $p = \sum_{k\neq 0} p_k e_{2k} = \Phi_m(q)$ and $n \geqslant M_m$, let

$$p^\circ = \sum_{0<|k|\neq n} p_k e_{2k}, \qquad q^\circ = \Phi_m^{-1}(p^\circ).$$

Then the $|n|$-th Fourier coefficients of $\Phi_m(q^\circ)$ vanish, which means that $\alpha_n(q^\circ)$ is a double periodic eigenvalue of q° of geometric multiplicity 2. Therefore,

$$\delta_n(q^\circ) = 0.$$

With $q^t = tq + (1-t)q^{\mathrm{o}}$, we get

$$\begin{aligned}\delta_n(q) &= \delta_n(q) - \delta_n(q^{\mathrm{o}}) \\ &= \int_0^1 \langle \mathrm{d}\delta_n(q^t), q - q^{\mathrm{o}} \rangle \,\mathrm{d}t \\ &= \langle t_n, q - q^{\mathrm{o}} \rangle + \langle \theta_n, q - q^{\mathrm{o}} \rangle\end{aligned}$$

with $\theta_n = \int_0^1 (\mathrm{d}\delta_n - t_n)(q^t)\,\mathrm{d}t$. Moreover,

$$q - q^{\mathrm{o}} = p - p^{\mathrm{o}} + \Theta_m(p - p^{\mathrm{o}}),$$

with $\Theta_m = \int_0^1 (D\Phi_m^{-1} - I)(\Phi_m(q^t))\,\mathrm{d}t$. Altogether we obtain

$$\delta_n(q) = \langle t_n, p - p^{\mathrm{o}} \rangle + \langle t_n, \Theta_m(p - p^{\mathrm{o}}) \rangle + \langle \theta_n, q - q^{\mathrm{o}} \rangle .$$

The identity $\langle t_n, p - p^{\mathrm{o}} \rangle = \kappa p_n + \bar{\kappa} p_{-n}$, the estimates

$$\|\theta_n\|_{\mathrm{o}} \leqslant \frac{C_m}{n} \leqslant \frac{1}{16}, \qquad \|\Theta_m\|_{L(\mathcal{H}^{\mathrm{o}},\mathcal{H}^{\mathrm{o}})} \leqslant \frac{1}{6}$$

by Lemma 6, as well as $\|t_n\|_{\mathrm{o}} \leqslant 1$ and $\|p - p^{\mathrm{o}}\|_{\mathrm{o}} \leqslant |p_n| + |p_{-n}|$ then give the claim. ∎

We now prove Theorem 4. Given $q \in B_m^w$ and assuming $n \geqslant N_m$, we have by the preceding lemma

$$|\delta_n(q)|^2 \leqslant (|p_n| + |p_{-n}|)^2 \leqslant 2\,|c_n|_{U_n}^2 + 2\,|c_{-n}|_{U_n}^2 .$$

We are thus in exactly the same situation as at the end of section 5, modulo a factor 4/9. So we get

$$\sum_{n \geqslant N} w_n^2\,|\delta_n(q)|^2 \leqslant 4\,\|T_N q\|_w^2 + \frac{256}{N}\,\|q\|_w^4$$

for all $N \geqslant N_m$, as well as

$$w_n\,|\delta_n(q)| \leqslant 4\,\|q\|_w$$

for all $n \geqslant N_m$. This establishes (i) of Theorem 4.

To prove the converse statement (ii), we only need to augment the proof of Theorem 3 in the case where p_n and p_{-n} are not about the same size. So suppose q

is in $\mathscr{H}^{\mathrm{o}}$ with

$$\sum_{n\geqslant 1} w_n^2 \left(|\gamma_n(q)| + |\delta_n(q)|\right)^2 < \infty.$$

Fix $m \geqslant 4\,\|q\|_{\mathrm{o}}$, and consider the coefficients $p_n = c_n(\alpha_n)$ for $|n| \geqslant M_m$. For any such n, for which the hypotheses of Lemma 10 are satisfied, we have

$$|p_n|\,, |p_{-n}| \leqslant 2\,|\gamma_n(q)|\,.$$

Otherwise, we may assume that $|p_n| \geqslant 4\,|p_{-n}|$, and we can use the preceding lemma to the effect that

$$\begin{aligned} |\delta_n(q)| &\geqslant |\kappa p_n + \bar{\kappa}\, p_{-n}| - \frac{1}{4}\left(|p_n| + |p_{-n}|\right) \\ &\geqslant \frac{1}{2}\cdot\frac{3}{4}\,|p_n| - \frac{1}{4}\cdot\frac{5}{4}\,|p_n| \\ &= \frac{1}{16}\,|p_n|\,. \end{aligned}$$

So in this case we get

$$|p_n|\,, |p_{-n}| \leqslant 16\,|\delta_n(q)|\,.$$

We again conclude that $p = \Phi_m(q) \in \mathscr{H}^w$, and the result follows with Proposition 8. This proves Theorem 4.

10 Superexponential Weights

We prove Theorem 6 by using the gap estimates already established for exponential weights. If $q \in \mathscr{H}^w$ with a superexponential weight w, then in particular $q \in \mathscr{H}^a$ for all $a \geqslant 0$, where a stands for the exponential weight $\exp(a\,|\cdot|)$. Given any $n \geqslant 4\,\|q\|_w$, we may thus choose

$$a = \psi(\tilde{n}) = \min_{m\geqslant 1} \frac{\log \tilde{n}\, w(m)}{m} \geqslant 0, \qquad \tilde{n} = \frac{n}{4\,\|q\|_w} \geqslant 1.$$

Then $\mathrm{e}^{am}/w_m \leqslant \tilde{n}$ for all $m \geqslant 1$, and consequently

$$\|q\|_a \leqslant \sup_{m\geqslant 1} \frac{\mathrm{e}^{am}}{w_m}\,\|q\|_w \leqslant \tilde{n}\,\|q\|_w = \frac{n}{4}.$$

We may thus apply the individual gap estimate given at the end of section 5 to obtain

$$|\gamma_n(q)| \leqslant \frac{6}{a_n} \|q\|_a \leqslant 2n\,\mathrm{e}^{-an} = 2n\,\mathrm{e}^{-n\psi(\tilde{n})}.$$

This is the claim, and Theorem 6 is proven.

Incidentally, the result is the same for auxiliary gap lengths, using the individual estimate given at the end of the preceding section. We only have to assume in addition that $n \geqslant N_m$, a constant depending only on $\|q\|_0$.

11 Extensions

L^p-spaces. For the sake of brevity and clarity we restricted ourselves to spaces $\mathcal{H}^w$ defined in terms of L^2-type norms. But we may also consider the spaces

$$\mathcal{H}^{w,r} = \left\{q = \textstyle\sum_{n\in\mathbb{Z}} q_n e_{2n} : \|q\|_{w,r} < \infty\right\}$$

for $1 \leqslant r \leqslant \infty$, where

$$\|q\|_{w,r}^r = \sum_{n\in\mathbb{Z}} w_n^r \, |q_n|^r\,, \qquad 1 \leqslant r < \infty,$$

$$\|q\|_{w,\infty} = \sup_{n\in\mathbb{Z}} w_n \, |q_n|\,.$$

The shifted norms $\|\cdot\|_{w,r;i}$ are defined analogously. The results remain the same, except for some minor quantitative aspects of constants and thresholds. The only new ingredient is an extended version of Lemma 1.

Lemma 1-R *If $q \in \mathcal{H}^{w,r}$ with $w \in \mathcal{M}$, then for $n \geqslant 1$ and $\lambda \in U_n$,*

$$T_n = VA_\lambda^{-1}Q_n$$

is a bounded linear operator on $\mathcal{H}_\star^{w,r}$ with norm $\|T_n\|_{w,r;i} \leqslant \dfrac{c_r}{n}\|q\|_{w,r}$ for all $i \in \mathbb{Z}$, where

$$c_r^s = \sum_{m\geqslant 1} \frac{2}{m^s}, \qquad s = \frac{r}{r-1}.$$

for $r > 1$ and $c_1 = 1$.

Proof. Consider the case $1 < r < \infty$. As in the proof of Lemma 1, we may write

$$g = A_\lambda^{-1} Q_n f = \sum_{|m| \neq n} \frac{f_m}{\lambda - m^2\pi^2} e_m = \sum_{m \in \mathbb{Z}} g_m e_m,$$

and by Hölder's inequality for $r^{-1} + s^{-1} = 1$ we get

$$\|ge_i\|_{w,1} \leqslant \|f\|_{w,r;i} \left(\sum_{|m| \neq n} \frac{1}{|m^2 - n^2|^s} \right)^{1/s}.$$

One verifies that

$$\sum_{|m| \neq n} \frac{1}{|m^2 - n^2|^s} \leqslant \frac{1}{n^s} \sum_{m \geqslant 1} \frac{2}{m^s} \leqslant c_r^s,$$

so that $\|ge_i\|_{w,1} \leqslant c_r \|f\|_{w,r;i}$. By standard estimates for the convolution of two sequences and the submultiplicity of the weights, one then arrives at

$$\|T_n f\|_{w,r;i} = \|Vg\|_{w,r;i} \leqslant \|V\|_{w,r} \|ge_i\|_{w,1} \leqslant c_r \|q\|_{w,r} \|f\|_{w,r;i}.$$

This holds for any $f \in \mathcal{H}^{w,r}$, so the claim follows for $1 < r < \infty$. The remaining cases are handled analogously. ∎

Subexponential weights. Our definition of a subexponential weight is chosen to allow for a convenient hypothesis of Lemma 9. But we might as well *define* a weight $w \in \mathcal{M}$ to be subexponential, if $\log w(n)/n \to 0$ and

$$\min(w, v_\varepsilon) \in \mathcal{M}$$

for all sufficiently small positive ε. Then Theorems 3 and 4 remain valid.

12 Epilog

[2] A preprint of this paper appeared in 2004 in the "Stuttgarter Mathematische Berichte", where it is still available. It is referenced by Djakov & Mityagin [5] at the end of section 3 with the remark "that Lemma 47 suggested in [68] plays important technical role later." They pretend that otherwise it just amounts to a "modified exposition of results from [38, 4, 6]" – which is [12, 2, 3] in this bibliography.

[2]This epilog was revised on May 1, 2008, due to legal threats by B. Mityagin.

In my view, this amounts to a gross distortion of the actual sequence of events, to put it mildly.

That lemma 47, or lemma 9 is this paper, is only a technical ingredient to apply the inverse function theorem to a certain functional in weighted Sobolev spaces. The lemma in itself is of no point without that approach in mind. And that approach was first described in the 2004 preprint, and was later adopted by Djakov & Mityagin.

The central idea developed in that preprint is to consider all objects, such as the coefficients a_n and c_n, as analytic functions of the potential q. This view never ever occurred in any of Dyakov & Mityagin's papers before 2004. In their expositions, q was always one fixed potential.

It is easy to check that their lemma 51 is a copy of Proposition 6 – but no credit is given. Likewise, their proof of proposition 53 is a remake of the proof of Proposition 8, and the reasoning for their theorem 54 is similarly taken from this paper. Again, no credit is given. It is also easy to check that none of this appeared in [12, 2, 3].

The 2004 preprint was also the first paper where converse regularity results were obtained for subexponential weights in general without any further qualifications, not just weights of a specific built such as Gevrey type weights. Comment 5.2 in [2] makes it clear that Dyakov & Mityagin had no proof of this general result. The same applies to their paper [3].

The second mention of the 2004 preprint in section 3.5 of [5] also suggests that I was using a trick from [59] to "balance a head and a tail". But this reference [59] is an unpublished manuscript not available in any archive and without any link. Anyway, the problem of proving the density of finite-gap potentials has no bearing on the inverse problem.

Finally, even a superficial perusal of [12, 2, 3] and related papers should make it clear that it takes more than the suggestion of a lemma and a "modified exposition" to cut away dozens of pages of clumsy inequalities and crappy exposition.

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