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On the Well-Posedness of the Periodic KdV Equation in High Regularity Classes

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1 Results

We consider the inital value problem for the periodic KdV equation,

$$u_t = -u_{xxx} + 6uu_x, \qquad u\big|_{t=0} = u_0, \tag{1}$$

where all functions are considered to be defined on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

According to one of the first results in this direction due to Bona & Smith [5] this problem has a unique, global solution for any initial value in one of the standard Sobolev space $\mathcal{H}^m = H^m(\mathbb{T}, \mathbb{R})$ with $m \geqslant 2$. That is, for each $u_0 \in \mathcal{H}^m$ there exists a unique continuous curve

$$\varphi\colon \quad \mathbb{R} \to \mathcal{H}^m, \quad t \mapsto \varphi(t, u_0)$$

solving the initial value in the sense defined below. Moreover, taken together they define a continous flow

$$\mathbb{R} \times \mathcal{H}^m \to \mathcal{H}^m, \quad (t, u_0) \mapsto \varphi(t, u_0).$$

Thus, the initial value problem is globally well-posed on $\mathcal{H}^m$ with $m \geqslant 2$ in the sense of Hadamard: solutions exist for all time, are unique, and depend continuously on their initial values.

Well-Posedness

Before we proceed we fix some notions. Let $\mathcal{H}^r = H^r(\mathbb{T}, \mathbb{R})$ be the usual Sobolev space of 1-periodic, real valued functions for real $r \geqslant 0$. A continuous curve $\varphi\colon I \to \mathcal{H}^r$ is called a *solution* of the initial value problem (1), if it solves (1) is the usual sense of distributions with $\varphi(0) = u_0$. It is called *global*, if $I = \mathbb{R}$.

We then say that the initial value problem (1) is *globally well-posed in* $\mathcal{H}^r$, if it has a global solution for each initial value in $\mathcal{H}^r$, and the resulting flow

$$\mathbb{R} \times \mathcal{H}^r \to \mathcal{H}^r, \quad (t, u) \mapsto \varphi(t, u)$$

is continuous. Moreover, we call (1) *globally uniformly well-posed in* $\mathcal{H}^r$, if it is globally well-posed, and for every compact interval I the map

$$\mathcal{H}^r \to C^0(I, \mathcal{H}^r), \quad u \mapsto \varphi(\cdot, u)$$

is uniformly continuous on bounded subsets of $\mathcal{H}^r$ with respect to the usual sup-norm on the second space. Well-posedness in the spaces $\mathcal{H}^w$ introduced later is defined analogously.

Known Results

Since the first results of Temam [31], Sjöberg [30] and Bona & Smith [5], the inital value problem for KdV and its well-posedness have been studied intensively. An excellent overview with a detailed bibliography is provided by the web site created by Colliander, Keel, Staffilani, Takaoka & Tao [11].

One focus has been on *low regularity solutions* in Sobolev spaces $\mathcal{H}^r$ with $r \leqslant 0$. We mention the works [6, 7, 8, 9, 10, 20, 21, 22]. As a result, KdV is now known to be globally well-posed in $\mathcal{H}^r$ for every $r \geqslant -1$, and globally uniformly well-posed in $\mathcal{H}^r$ for every $r \geqslant -1/2$. Incidentally, it is an interesting phenomenon, that an equation can be globally well-posed, but not in a uniform way.

In this paper we focus on *high regularity solutions*. These are solutions in a general class of weighted Sobolev spaces within $\mathcal{H}^0$, that encompass analytic and Gevrey spaces, among others. Some results in this direction on the *real line* can be found in [4, 14]. But in general, the question of existence and well-posedness of solutions of nonlinear pdes of high regularity have not been widely considered. We this that this topic deserves to be studied in more depth, revealing important features of the nonlinear equation considered.

Weighted Sobolev spaces

To state our results we first introduce *weighted Sobolev spaces* within the standard space

$$\mathcal{H}^0 = L^2(\mathbb{T})$$

of square-integrable functions $u = \sum_{n\in\mathbb{Z}} u_n e^{2\pi i n x}$. As in [16, 17] a *weight* is a *normalized*, *symmetric* and *submultiplicative* function $w\colon \mathbb{Z} \to \mathbb{R}$. That is, for all integers n and m, we have

$$w_n \geqslant 1, \qquad w_{-n} = w_n, \qquad w_{n+m} \leqslant w_n w_m.$$

We then define the *weighted Sobolev spaces*

$$\mathcal{H}^w := \left\{ u \in \mathcal{H}^0 : \|u\|_w^2 := \sum_{n\in\mathbb{Z}} w_n^2 \, |u_n|^2 < \infty \right\}.$$

To give some examples, let $\langle n \rangle = 1 + |n|$. The *Sobolev weights* $\langle n \rangle^r$, $r \geqslant 0$, give rise to the usual Sobolev spaces $\mathcal{H}^r$ of 1-periodic, complex-valued functions. In particular, for nonnegative integers m we obtain the standard spaces $\mathcal{H}^m$. The *Abel weights*[1] $\langle n \rangle^r e^{a|n|}$ with $a > 0$ define spaces $\mathcal{H}^{r,a}$ of functions in $\mathcal{H}^r$, which are analytic on the complex strip $|\operatorname{Im} z| < a/2\pi$ and have traces in $\mathcal{H}^r$ on the boundary lines. The *Gevrey weights*

$$w_n = \langle n \rangle^r e^{a|n|^\sigma}, \qquad r \geqslant 0,\ a > 0,\ 0 < \sigma < 1,$$

lie in between and give rise to the so called Gevrey spaces $\mathcal{H}^{r,a,\sigma}$ of smooth 1-periodic functions. Obviously,

$$\mathcal{H}^{r,a} = \mathcal{H}^{r,a,1} \subsetneq \mathcal{H}^{r,a,\sigma} \subsetneq \mathcal{H}^{r,a,0} = \mathcal{H}^r.$$

Since $\log w_n$ is subadditive and nonnegative, the limit

$$\chi(w) := \lim_{n\to\infty} \frac{\log w_n}{n}$$

exists and is nonnegative [29, no. 98]. Naturally, we call a weight w *exponential*, if $\chi(w) > 0$. We call w *subexponential*, if $\chi(w) = 0$ *with* $\log w_n / n$ converging to

[1] The term *Abel* weights is chosen to go along with *Sobolev* and *Gevrey* weights and has no deeper meaning.

zero in an eventually monotone manner. This is not an exact dichotomy, but we are not aware of any interesting weight that does not belong to either class. Clearly, Abel weights are exponential, while Sobolev and Gevrey weights are subexponential.

Theorems

Theorem 1 *The periodic KdV equation is globally uniformly well-posed in every space $\mathcal{H}^w$ with a subexponential weight w. That is, for each initial value u in one of these spaces $\mathcal{H}^w$ the associated Cauchy problem has a global solution $t \mapsto \varphi^t(u)$ in $\mathcal{H}^w$, giving rise to a continuous flow*

$$\mathbb{R} \times \mathcal{H}^w \to \mathcal{H}^w, \quad (t,u) \mapsto \varphi^t(u),$$

which is even uniformly continuous on bounded subsets of $\mathcal{H}^w$.

Indeed, the flow map is even analytic, see also [3]. For exponential weights the result is not as clear cut.

Theorem 2 *The periodic KdV equation is »almost« globally well-posed in every space $\mathcal{H}^w$ with an exponential weight w. That is, for each bounded subset $\mathcal{B}$ of $\mathcal{H}^w$ there exists $0 < \rho \leqslant 1$ such that the Cauchy problem for each initial value $u \in \mathcal{B}$ has a global solution $t \mapsto \varphi^t(u)$ in $\mathcal{H}^{w^\rho}$, giving rise to a continuous flow*

$$\mathbb{R} \times \mathcal{B} \to \mathcal{H}^{w^\rho}, \quad (t,u) \mapsto \varphi^t(u).$$

Here, w^ρ is the weight with $(w^\rho)_n = w_n^\rho$, which is again normalized, symmetric and submultiplicative. Thus, for initial values u in a bounded subset $\mathcal{B}$ of $\mathcal{H}^{0,a}$, say, (1) has a global solution in $\mathcal{H}^{0,\rho a}$ with a fixed $0 < \rho \leqslant 1$. It is an open question, whether ρ can be chosen to be 1. For related results, see for example [1].

These results are not restricted to the standard KdV equation, but apply simultaneously to all equations in the KdV hierarchy, as defined for instance in [18]. The second KdV equation, for example, reads

$$u_t = u_{xxxxx} - 10uu_{xxx} - 20u_x u_{xx} + 30u^2 u_x.$$

Such a hierarchy may be defined in a variety of ways, but this is immaterial here and does not affect the statement of the following theorem.

Theorem 3 *Theorems 1 and 2 also hold for every KdV equation in the KdV hierarchy, provided that in the case of Sobolev spaces $\mathcal{H}^r$, r is sufficiently large.*

Our results naturally extend the KAM theory of Hamiltonian perturbations of KdV equations developed by Kuksin [23, 24, 25] and expounded in [26, 18]. Consider the perturbed KdV equation

$$\frac{\partial u}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial H}{\partial u} + \varepsilon \frac{\partial K}{\partial u}\right).$$

If K is real analytic in u with a gradient $\partial K/\partial u$ in some standard Sobolev space $\mathcal{H}^m$, $m \geqslant 1$, then KAM for KdV asserts the persistence of *quasi-periodic* solutions for sufficiently small $\varepsilon \neq 0$. Theorems 1 and 2 may now be extended as follows – for a more precise statement we refer to [19].

Theorem 4 *Under sufficiently small Hamiltonian perturbations, the majority of the quasi-periodic solutions of the KdV equation persists, their regularity being only slightly less than the regularity of the perturbing term.*

These theorems are based on two observations. First, the periodic KdV equation is well known to be an *infinite dimensional, integrable Hamiltonian system*. As such, it even admits global Birkhoff coordinates $(x_n, y_n)_{n\geqslant 1}$ defined as the cartesian counterpart to global action angle coordinates $(I_n, \theta_n)_{n\geqslant 1}$. Second, there is a precise correspondence between the decay properties of the coordinates $(x_n, y_n)_{n\geqslant 1}$ and the regularity properties of u. The link is provided by the spectral properties of the associated Hill operator

$$L_u = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u$$

on the interval $[0, 2]$ with periodic boundary conditions.

In the rest of this note we describe this approach in more detail, but without lengthy proofs. These are given in [19].

2 Birkhoff Coordinates

As is well known, the KdV equation can be written as an infinite dimensional Hamiltonian system

$$\frac{\partial u}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial H}{\partial u}$$

with Hamiltonian

$$H(u) = \int_{\mathbb{T}} \left(\tfrac{1}{2}u_x^2 + u^3\right) \mathrm{d}x.$$

As a phase space one may take

$$\mathcal{H}_0^m = \left\{ u \in \mathcal{H}^w : [u] := \textstyle\int_{\mathbb{T}} u \,\mathrm{d}x = 0 \right\}$$

with $m \geqslant 1$, as the KdV flow preserves mean values. The Poisson bracket proposed by Gardner,

$$\{F, G\} = \int_{\mathbb{T}} \frac{\partial F}{\partial u} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial G}{\partial u} \,\mathrm{d}x,$$

then makes $\mathcal{H}_0^m$ into a nondegenerate Poisson manifold, such that $u_t = \{u, H\}$.

Next, we introduce the weighted sequence spaces

$$h^w = \ell^w \times \ell^w$$

with elements (x, y), where

$$\ell^w = \left\{ x = (x_n)_{n \geqslant 1} : \|x\|_w^2 = \textstyle\sum_{n \geqslant 1} w_n^2 |x_n|^2 < \infty \right\}.$$

We endow h^w with the standard Poisson structure, for which $\{x_n, y_m\} = \delta_{nm}$, while all other brackets vanish. To simplify notations, we further introduce

$$h_\star^w = \ell_\star^w \times \ell_\star^w, \qquad \ell_\star^w = \left\{ x \in \ell^w : (\sqrt{n} x_n)_{n \geqslant 1} \in \ell^w \right\}.$$

The extra weight $\sqrt{n}$ reflects the effect of the derivative $\mathrm{d}/\mathrm{d}x$ in the Gardner bracket.

The following theorem was first proven in [2, 3]. A quite different approach was first presented in [15], and a comprehensive exposition is given in [18]. Note that $\mathcal{H}_0^0 = \left\{ u \in L^2(\mathbb{T}) : [u] = 0 \right\}$.

Theorem 5 *There exists a diffeomorphism*

$$\Omega : \ \mathcal{H}_0^0 \to h_\star^0$$

with the following properties.

(a) *Ω is onto, bi-analytic, and takes the standard Poisson bracket into the Gardner bracket.*

(b) *The restriction of Ω to $\mathcal{H}_0^m$, $m \geqslant 1$, gives rise to a map $\Omega : \mathcal{H}_0^m \to h_\star^m$, which is again onto and bi-analytic.*

(c) *Ω introduces global Birkhoff coordinates for the KdV Hamiltonian on $\mathcal{H}_0^1$. That is, on $h_\star^1$ the transformed KdV Hamiltonian $H \circ \Omega^{-1}$ is a real analytic function of*

$$I_n = \tfrac{1}{2}(x_n^2 + y_n^2), \qquad n \geqslant 1.$$

(d) *The last statement also applies to every other Hamiltonian in the KdV hierarchy, if '1' is replaced by 'm' with m sufficiently large.*

Denoting the transformed KdV Hamiltonian by the same symbol we thus obtain a real analytic Hamiltonian

$$H = H(I_1, I_2, \dots)$$

on $h_\star^1$. Its equations of motion are the classical ones,

$$\dot{x}_n = H_{y_n}, \quad \dot{y}_n = -H_{x_n}, \qquad n \geqslant 1,$$

since the Poisson structure on $h_\star^1$ is the standard one. It is therefore evident that every solution of the KdV equation exists for all time, and is indeed *almost periodic*. More precisely, every solution winds around some underlying invariant torus

$$T_I = \prod_{n \geqslant 1} S_{I_n}, \qquad S_{I_n} = \left\{ x_n^2 + y_n^2 = 2I_n \right\},$$

which is fixed by the actions of the initial position. The speed on the n-th circle S_{I_n} is determined by the n-th frequency

$$\omega_n = H_{I_n}(I_1, I_2, \dots),$$

and the entire flow is given by

$$\psi^t(x, y) = (x_n \cos \omega_n t, y_n \sin \omega_n t)_{n \geqslant 1}.$$

Obviously, ψ^t preserves all weighted norms and thus all weighted spaces $h_\star^w$.

To obtains our results about the well-posedness of the KdV equation, we now formulate two extensions of Theorem 5. First we consider subexponential weights.

Theorem 6 *For each subexponential weight w, the restriction of Ω to $\mathcal{H}_0^w$ gives rise to an onto, bi-analytic diffeomorphism $\Omega : \mathcal{H}_0^w \to h_\star^w$.*

Proof of Theorem 1. Due to its symplectic nature, Ω maps solution curves $t \mapsto \varphi^t(u)$ in function space into solution curves $t \mapsto \psi^t(x, y)$ in sequence space with $(x, y) = \Omega(u)$. Since Ω is also a diffeomorphism between $\mathcal{H}_0^w$ and $h_\star^w$ and ψ^t preserves $h_\star^w$, the diagramm

$$\begin{array}{ccc} u \in \mathcal{H}_0^w & \xrightarrow{\ \Omega\ } & (x, y) \in h_\star^w \\ \varphi^t \big\downarrow & & \big\downarrow \psi^t \\ \varphi^t(u) \in \mathcal{H}_0^w & \xleftarrow{\ \Omega^{-1}\ } & \psi^t(x, y) \in h_\star^w \end{array}$$

is correct and proves the theorem. ∎

Now we consider exponential weights. Here, the result is not as elegant.

Theorem 7 *Let w be an exponential weight. Then for every bounded subset B of $h_\star^w$ there exists $0 < \rho \leqslant 1$ such that $\Omega^{-1}(B) \subset \mathcal{H}_0^{w^\rho}$.*

Proof of Theorem 2. Let w be an exponential weight and $\mathcal{B}$ a bounded subset of $\mathcal{H}_0^w$. Then $B = \Omega(\mathcal{B})$ is a bounded subset of $h_\star^w$ by Proposition 8 below. As the flow ψ^t preserves the $h_\star^w$-norm, the set

$$B^- = \bigcup_{t \in \mathbb{R}} \psi^t(B)$$

is contained in the same centered ball as B. Hence, by the previous theorem there exists a $0 < \rho \leqslant 1$ such that $\mathcal{B}^- = \Omega^{-1}(B^-)$ is contained in $\mathcal{H}_0^{w^\rho}$. We obtain the commutative diagramm

$$\begin{array}{ccc} \mathcal{B} \subset \mathcal{H}_0^w & \xrightarrow{\ \Omega\ } & B \subset h_\star^w \\ \varphi^t \big\downarrow & & \big\downarrow \psi^t \\ \mathcal{B}^- \subset \mathcal{H}_0^{w^\rho} & \xleftarrow{\ \Omega^{-1}\ } & B^- \subset h_\star^w \end{array}$$

which proves the theorem. ∎

Proof of Theorem 3. The proofs of Theorem 1 and 2 are based on the fact that the map Ω triviales the KdV flow in the Birkhoff coordinates. By item (iv) of Theorem 5, however, Ω simultaneously trivializes any other KdV flow in the KdV hierarchy. The only difference is in the frequencies ω_n associated with the circles S_{I_n}, and in the minimal regularity required for the KdV hamiltonians to make sense. Hence the preceding proofs apply to higher KdV equations as well. ∎

3 Regularity

Theorems 6 and 7 are based on two observations. First, the asymptotics of the Birkhoff coordinates of a function u in $\mathcal{H}_0^0$ are closely related to the asymptotics of its spectral gaps. Second, these asymptotics are very closely related to the regularity of u. To keep the discussion simple, we restrict ourselves to the real case.

Spectral Gaps and Actions

For a *potential* $u \in L_0^2 = \mathcal{H}_0^0$ consider Hill's operator

$$L_u = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u$$

on the interval $[0, 2]$ with periodic boundary conditions. As is well known, its spectrum is pure point and consists of an unbounded sequence of real eigenvalues

$$\lambda_0(u) < \lambda_1^-(u) \leqslant \lambda_1^+(u) < \lambda_2^-(u) \leqslant \dots \ .$$

Its so called *spectral gaps* are the – possibly empty – intervals $(\lambda_n^-(u), \lambda_n^+(u))$, and one speaks of the *gap lengths* $\gamma_n(u) = \lambda_{2n}(u) - \lambda_{2n-1}(u)$ of u.

Proposition 8 ([18, p. 67]) *There exists a complex neighbourhood W of L_0^2 such that each quotient I_n/γ_n^2 extends analytically to W and satisfies*

$$8\pi n \frac{I_n}{\gamma_n^2} = 1 + O\left(\frac{\log n}{n}\right), \qquad n \geqslant 1,$$

locally uniformly on W, as well as uniformly on bounded subsets of L_0^2.

So we have

$$n(x_n^2 + y_n^2) \sim n I_n \sim \gamma_n^2$$

locally uniformly on W. This gives us control of $x_n^2 + y_n^2$ in terms of γ_n^2 on the *real* space L_0^2, where all quantities are *real*. This is not the case on the *complex* neighbourhood W, where a gap γ_n and thus an action I_n may vanish, while the Birkhoff coordinates x_n, y_n do *not*. See [19] for the details of this case.

Spectral Gaps and Regularity

The decay properties of spectral gaps are also closely tied to the regularity of the potential. For example, a classical result due to Marčenko & Ostrowskiĭ [27] states that

$$u \in \mathcal{H}^m \quad \Leftrightarrow \quad \sum_{n \geqslant 1} n^{2m} \gamma_n^{2m}(u) < \infty$$

for any integer $m \geqslant 0$. The forward part of this result generalizes as follows.

Theorem 9 *For any subexponential or exponential weight w,*

$$u \in \mathcal{H}^w \quad \Rightarrow \quad (\gamma_n(u)) \in \ell^w.$$

Consequently, for any subexponential or exponential weight w, the Birkhoff map Ω maps $\mathcal{H}_0^w$ *into* $\ell_\star^w$. Indeed,

$$u \in \mathcal{H}^w \quad \Rightarrow \quad (\gamma_n) \in \ell^w \quad \Rightarrow \quad (nI_n) \in \ell^w \quad \Rightarrow \quad (x_n, y_n) \in \ell_\star^w.$$

by Theorem 9 and Proposition 8.

A one-to-one converse to this theorem is only true in the subexponential case

Theorem 10 ([12, 28]) *For a subexponential weight w,*

$$(\gamma_n(u)) \in \ell^w \quad \Rightarrow \quad u \in \mathcal{H}^w.$$

Consequently, in the subexponential case, the Birkhoff map Ω also maps $\mathcal{H}_0^w$ *onto* $\ell_\star^w$. Indeed,

$$\begin{aligned} & (x, y) \in \ell_\star^w \subset \ell_\star^0 \\ \Rightarrow \quad & u = \Omega^{-1}(x, y) \in \mathcal{H}_0^0 \quad \textit{with} \quad \gamma_n^2 \sim n(x_n^2 + y_n^2) \\ \Rightarrow \quad & (\gamma_n(u)) \in \ell^w \\ \Rightarrow \quad & u \in \mathcal{H}^w. \end{aligned}$$

Altogether, Ω is a diffeomorphism between $\mathcal{H}_0^w$ and $\ell_\star^w$ whenever w is a subexponential weight. Thus, Theorem 6 is proven.

The last theorem does not extend to exponential weights, however. This is exemplified by finite gap potentials such as the Weierstrass $\wp$-function, which are not entire functions. Gasymov [13] even observed that *any* complex potential of the form

$$u = \sum_{n \geqslant 1} u_n e^{2\pi i n x} = \sum_{n \geqslant 1} u_n z^n \Big|_{z = e^{2\pi i x}}$$

is a 0-gap-potential. So in the complex case, the gap sequence need not contain *any* information about the regularity of the potential.

In the real case, however, we have the following classical result by Trubowitz. The very last statement is proven in [28].

Theorem 11 ([32]) *For an exponential weight w,*

$$(\gamma_n(u)) \in \ell^w \quad \Rightarrow \quad u \in \mathcal{H}^{w^\rho},$$

where $0 < \rho \leqslant 1$ depends on $\|u\|_{L^2}$ and $\sum_n w_n^2 \gamma_n^2$.

Consequently, for any bounded subset B of $\ell_\star^w$ there exists $0 < \rho \leqslant 1$ so that

$$\Omega^{-1}(B) \subset \mathcal{H}_0^{w^\rho}.$$

Indeed, $A = \Omega^{-1}(B)$ is bounded in L_0^2, and by Proposition 8,

$$\sum_{n \geqslant 1} w_n^2 |\gamma_n^2(u)| \leqslant c \sum_{n \geqslant 1} n w_n^2 \left(|x_n^2(u)| + |y_n^2(u)|\right)$$

uniformly on A. The latter sum is uniformly bounded by assumption, so by Theorem 11 we have $A \subset \mathcal{H}_0^{w^\rho}$ for some $0 < \rho \leqslant 1$. This establishes Theorem 7.

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